SPOTLIGHT ON THE STANDARDS

JENNIFER M. BAY-WILLIAMS AND SHERRI L. MARTINIE

Thinking Rationally about Number and Operations in the Middle School

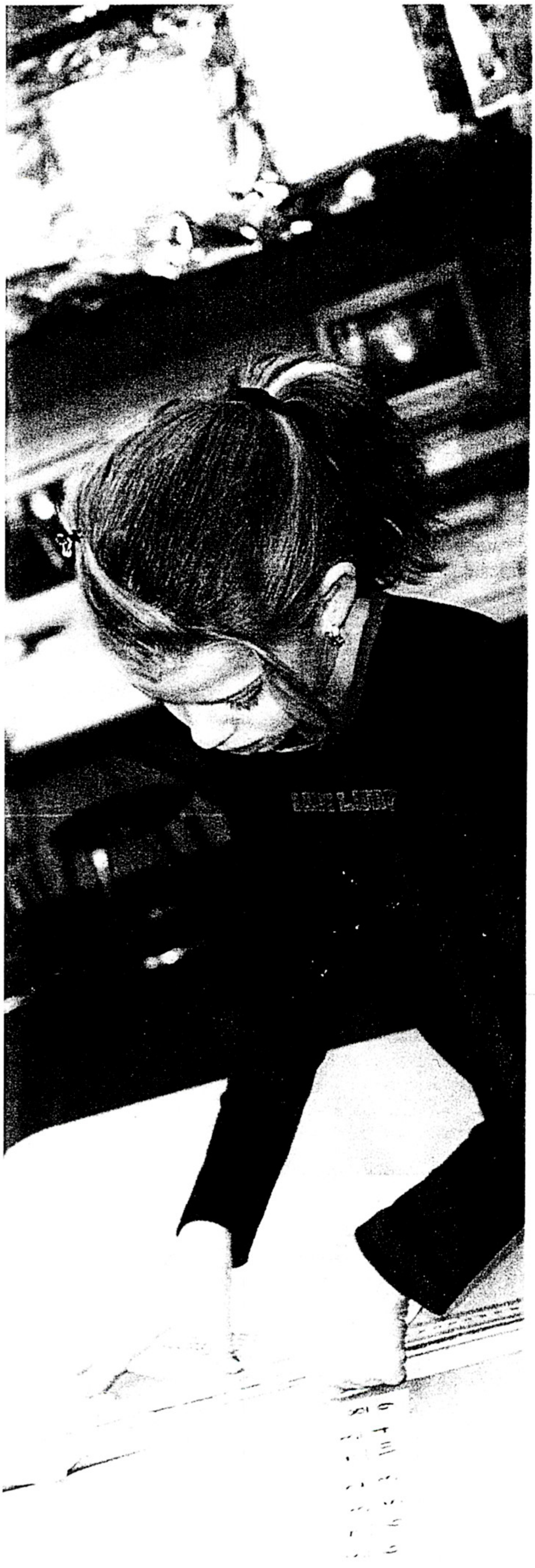
CONSIDER THE FOLLOWING THREE SITUATIONS:

1. Suppose that Julianna and Fran are racing, and Julianna has completed 3/5 of the race while Fran has completed 2/3 of the race. Who is in the lead?
2. How many bows that are each 5/12 yards long can be made from 3 yards of ribbon?
3. If you were as strong (for your size) as an ant, how much weight could you lift? (Note: ants can lift 1/5 of an ounce, although they weigh only about 1/250 of an ounce.)

These three scenarios, which involve the relative size of fractions, fraction computation, and proportional reasoning, respectively, illustrate some of the middle school content expectations within the Number and Operations strand of *Principles and Standards for School Mathematics* (NCTM 2000). A facility with rational numbers "should be developed through experience with many problems involving a range of topics" (NCTM 2000, p. 212). In the following sections, we describe classroom experiences with the problems mentioned above and discuss other problems, as well. Each of these problems is "intriguing, with a level of challenge that invites speculation and hard work," in other words, "worthwhile tasks," as defined in *Principles and Standards for School Mathematics* (p. 19). We hope these classroom experiences serve to illuminate both the content emphasis of the middle-grades number strand and the focus on reasoning and problem solving that is important to the study of mathematics in general.

Understanding Rational Numbers

ONE CONTENT EXPECTATION FOR MIDDLE school students is to "compare and order fractions, decimals, and percents efficiently and find their approximate locations on a number line" (NCTM 2000, p. 393). We have found that even when students can determine the larger of two fractions or two decimals, they often do not have a sense of the size of that value when it is related to 0, 1/2, and 1 (Martinie and Bay-Williams, in press). In a fifth/sixth-grade classroom, the following problem was posed:



Red Light, Green Light

At recess, a group of students was playing "red light, green light." Mrs. Hancock was the caller and was very quick to catch students moving and send them back. After 10 minutes, no one had won, but each student was this fraction of the way to Mrs. Hancock:

Abilene: 1/4
Bob: 8/9
Corey: 7/10
Danielle: 9/21
Edwardo: 7/16
Fran: 2/3
Gina: 2/47
Henry: 5/6
Isabelle: 3/4
Julianna: 3/5
Kendall: 1/13
Leonardo: 1/8

Using only mental math, can you order these students from the start to the finish? Explain briefly how you placed each one.

The popular game "red light, green light" was used to provide an intriguing context that incorporated a number-line model for thinking about fractions.

The class began with five student volunteers actually playing "red light, green light." When the volunteers were "frozen," we asked observing students to describe approximately what fraction of the way each student was from the start to the finish line.

JENNIFER BAY-WILLIAMS, *jbay@ksu.edu, teaches undergraduate and graduate mathematics education courses at Kansas State University, Manhattan, KS 66506. Her interests include developing conceptual understanding in mathematics and using problem-based curriculum.* SHERRI MARTINIE, *martinies@usd320.k12.ks.us, teaches at Wamego Middle School, Wamego, KS 66547-9673. Martinie is interested in how learners—in particular, middle school students—come to understand and flexibly use rational numbers.*

Edited by DIANA V. LAMBDIN, Indiana University, Bloomington, IN 47405-1006

"Spotlight on the Standards" focuses on the grades 6–8 content and process standards found in NCTM's Principles and Standards for School Mathematics *(2000). The articles compare NCTM's* Curriculum and Evaluation Standards for School Mathematics, *published in 1989, with the* Principles and Standards *relating to the middle grades and suggest ways that teachers might incorporate Standards-based practices into their instruction.*

Students used many approaches to order the players. Some students used benchmarks, placing 0, 1/4, 1/2, 3/4, and 1 on the number line and making decisions about whether each of the fractional values in the problem was greater than or less than each of these values (see **fig. 1**).

Other students relied on their understanding of decimals. For example, one group converted all fractions to decimals (mentally), using familiar equivalencies to generate unknown ones. For example, to figure out 5/6, they knew that 3/6 = 1/2 = 0.50. Then they divided 50 by 3 to find 1/6, then multiplied by 5 to find 5/6 (see **fig. 2**). Some students used a blend of strategies, depending on the fractions, as **figure 3** explains. Several groups used centimeters or inches to determine the fractional amounts. A few students created one number line and then made slash marks for each denominator, erasing and redrawing. For fourths, they separated the number line into four equal sections and placed 1/4 and 3/4; for 3/47, they drew 47 tick marks and found 3/47. As students shared these varied solution strategies, they continue to develop their understanding of the relative size of fractions.

Some of the solution strategies described here indicate the flexible use of fractions. Flexibility with rational numbers includes (1) working flexibly within a form (i.e., among fractions only); (2) moving among fractions, decimals, and percents; and (3) knowing which form is most useful, given the problem at hand.

Moving among forms is important, but just as important is knowing when a particular form is most useful. In this particular class, many students had become overreliant on decimal equivalencies for comparing fractions. This situation became obvious when we posed the following basketball problem to them:

> In the middle of a close basketball game, our opponents are charged with a technical foul. Our coach gets to pick who she feels is the best free-throw shooter on our team. She looks at her record book

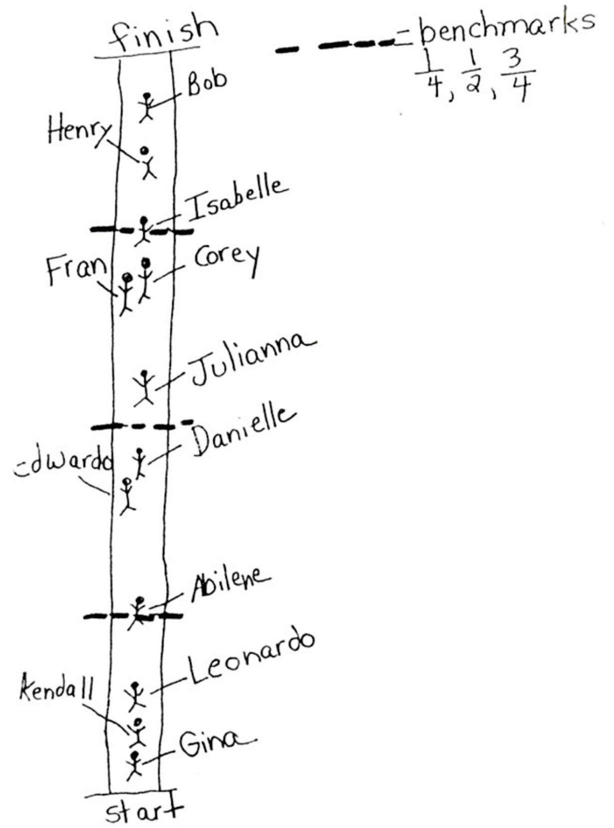


Fig. 1 A solution that used benchmarks to order the fractions

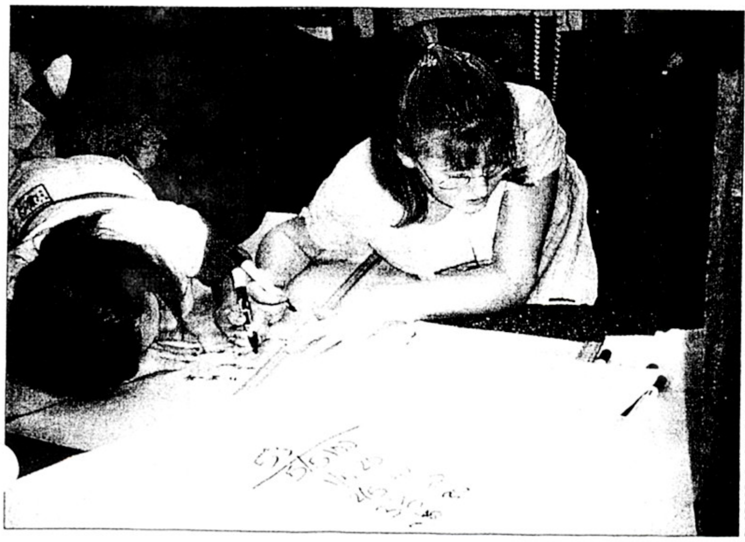


Fig. 2 Students use decimal equivalence for fifths and sixths to help order fractions.

I knew the smallest fraction was 2/47. I also knew 1/13 and 1/8 were both smaller than 1/4, and I knew 1/13 is smaller, because if you divide an object into 13 pieces those pieces will be smaller than if you divide an object of the same size into 8 pieces. Since both of these fractions are under 1/4, 1/4 is larger. The next fractions were 9/21 and 7/16. I knew they were both just under 1/2, because 1/2 of 21 = 10.5 and 1/2 of 16 = 8, since they were so close I decided 7/16 seems larger. I knew the next 4 decimals for the fractions, and put them in the order of 3/5 (.6), 2/3 (.66), 7/10 (.7), 3/4 (.75). Then, I had to decide if 5/6 or 8/9 was larger, because I didn't know the decimal for 5/6. So, I made them equivalent denominators. I multiplied 6×9 = 54 and 5×9 = 45 to get the fraction 45/54 = 5/6 and I multiplied 9×6 = 54 and 8×6 = 48 to get the fraction 48/54 = 8/9, so 5/6 is less than 8/9. So, the order was 2/47 1/13, 1/8, 1/4, 9/21, 7/16, 3/5, 2/3, 7/10, 3/4, 5/6, the 8/9.

Fig. 3 An explanation that incorporates decimal equivalencies to order fractions

and sees that Angela has made 17 of 25 free throws, Emily has made 15 of 20 free throws, and Carma has made 7 out of 10 free throws. Which players should our coach pick to shoot the free throw? (Adapted from Lappan et al. 1996)

Students quickly reached for their calculators and divided each number of shots made by the number of shots taken, found the decimals, and multiplied them by 100 to compute the respective percents. Although this approach demonstrated fluency in rewriting fractions as decimals and as percents, we were hoping for more thoughtful reasoning. We asked students to reconsider the problem, given that the coach was unlikely to have a calculator. As seen in **figure 4,** this prompt generated more reasoning in solving the problem.

Students need a lot of different experiences to help them develop the ability to work flexibly and thoughtfully with problems involving fractions, decimals, and percents. Students who have a deep understanding of frequently used fractions, decimals, and percents are equipped with knowledge that supports their understanding of the relative sizes of less frequently used rational numbers.

The Meaning of Operations

STUDENTS SHOULD UNDERSTAND THE *MEANING* and *effects* of arithmetic operations with rational numbers and should be able to *develop* and *analyze* algorithms when computing rational numbers. This focus on understanding operations with rational numbers, then applying this understanding to developing algorithms, is a major shift from a teaching model in which procedures for fraction computation are presented to students without examining how they work.

The following problem, involving a whole number divided by a fraction, was posed to a class of seventh graders:

Maria has 3 yards of ribbon that must be cut into sections of 5/12 yard each to make bows. How many bows can be made?

Because they had already encountered a number of other contexts that involved fractional divisors, some students were beginning to take shortcuts and develop an algorithm. For example, one student explained that for each yard, envision 12 parts (each 1/12 of a yard in length), so the total number of parts is 12 × 3, or 36. Each bow requires 5 of the parts, so the 36 parts need to be combined together in groups of 5. From 36 parts, 7 bows can be made, since 36 ÷ 5 = 7 (with some ribbon left over). Other students were more comfortable

First of all he'd know Emily has more than Carma. If you multiply 7×2 you get 14 and if you multiply 10×2 you get 20, so Carma's fraction is 14/20 and you know Emily's is 15/20, so you know Emily's is more than Carma's, so now you just have to know if Emily's is more than Angela's. I know (and the coach probably knows) 15/20 is the same as 3/4. I also know 1/4 of 24 is 6, so 3/4 is 18, and if 18/24 is 3/4 17/25 has to be less than 3/4, so that means that Emily has more than Angela since Emily has 3/4. Since Emily has more than anybody else, that means he should pick Emily.

Fig. 4 A solution to the second attempt at the free-throw-shooting problem

using a concrete model and solved this problem using Cuisenaire rods or pictures.

After multiple experiences with contextualized problems involving division of a whole number by a fraction (e.g., $c \div a/b$), students will begin to see that the denominator of the fractional divisor (b) defines how many parts to split *each* whole. Therefore, the total number of parts is (bc). The numerator of the fractional divisor (a) tells the group size, that is, how many parts must be in each group. To find how many groupings are possible, a student must divide the total number of parts (bc) by the group size (a). The standard algorithm for division by a fraction—"multiply by the reciprocal"—begins to emerge and make sense to students.

Few middle school students (or adults) find it easy to describe real-world situations that represent operations with fractions. In a study of U.S. and Chinese elementary teachers, Ma (1999) asked them to consider this division of fractions task:

$$1\ 3/4 \div 1/2$$

She asked each teacher to complete the computation and to suggest a real-world context in which this computation would be used. U.S. teachers were

Ants are small and mighty! One ant weighs about 1/250 of an ounce! It can lift a bread crumb that weighs about 1/5 of an ounce. If you were this strong, how much could you lift?

Fig. 5 The Ant Problem

Ant Problem

1. $\begin{array}{r} 50 \\ 5\overline{)250} \\ \underline{25} \\ 00 \end{array}$ 5/250 50/250

2. Aunt = 1/250 ounces
Bread = 1/5 ounces
= 50 times its weight

3. $\boxed{\text{my weight}}$ = 98 lbs. $\begin{array}{r} 98 \\ \times 50 \\ \hline 4900 \end{array}$ lbs.

4. I could lift two cars!

Fig. 6 One group's solution to the Ant Problem

ant Question

1/250 ant's Weight
1/5 lifts (once)

$\frac{1.0}{250} \div 10 = \frac{.1}{25} \times 4 = \frac{.4}{100}$

$\frac{1}{5} \times 2 = \frac{20}{100} - \frac{.4}{100} = \frac{19.6}{100}$

$\frac{.4}{100} \times 50 = \frac{20}{100}$

$100 \times 50 = 5{,}000 = 2\frac{1}{2}$ tons

EX: Can lift a Semi, Combine, a trailer, 10 cars, or a double decker buses.

Fig. 7 A second strategy for solving the Ant Problem

unable to adequately explain the computation (although some did remember the invert-and-multiply algorithm), and were also unable to provide appropriate real-world applications. If a person is unable to link an algorithm to a context, knowing the algorithms is of little use. By connecting contexts to algorithms, students will be more likely to select and apply the appropriate algorithms when confronted with new situations.

Proportionality

CONSIDERED A UNIFYING THEME IN MIDDLE-grades mathematics, proportional reasoning is a "key mathematical idea that permeates the curriculum" (Silver 2000, p. 22). Proportional reasoning is different from instructing students to set up proportions in a prescribed manner and apply a cross-multiplication algorithm to solve the proportion. Students need the opportunity to compare two related quantities, reason about the relationship between them, and develop their own methods of comparison.

The book *If You Hopped Like a Frog* by David Schwartz (1999) describes a series of proportional relationships between animals and people. Before reading the book, students were given task cards like the one shown in **figure 5.**

Figure 6 presents one of two different approaches to solving the problem. In this approach, the students were trying to figure out how many 5s are in 250. Once they figured out that the change factor was 50, they multiplied their own weight by 50. A 98-pound student determined that she would be able to lift 4,900 pounds!

A second group (see **fig. 7**) found common denominators for the ant's weight and the breadcrumb weight to compare the two fractions. With a common denominator of 100, the numerators of the two fractions are 0.4 and 20, respectively. At first, this group looked at the difference in the numerators (19.6). After some discussion, another student suggested that they find how many *times* its body weight an ant can lift. After guessing and checking, students figured out that $0.4 \times 50 = 20$. One student summarized, "This means we can lift that much more than what we weigh . . . 50 times."

As students discussed how to compare animal characteristics with their own, they were challenged to make sense of what is and what is not appropriate proportional reasoning. After solving other tasks, they listened to the story. Students were amazed by the exceptional qualities of the animals presented in the book and were better able to appreciate the mathematics contained in the text of the book.

Number and Operations in the Middle School

FROM ANT STRENGTH TO BOW CUTTING, FROM basketball games to backyard games, many contexts can be found that intrigue middle school students and engage them in the study of rational numbers. Embedding problems in contexts is essential if students are to understand and be able to use rational numbers effectively. Although some middle school students may still need continued work with whole-number concepts and computation, the focus of the middle school curriculum must be on rational numbers. Time must be devoted to exploring and developing a conceptual understanding. Middle school experiences related to number should focus on reasoning, promote flexible use of numbers, and incorporate multiple contexts and a variety of tools. When given a context for a problem, students are able to reason in ways that might not be possible when exploring context-free problems. Using various contexts to develop algorithms enables students to be more successful in solving problems involving operations with rational numbers (Irwin 2001). These experiences will enable students to make sense of and use fractions, decimals, and percents thoughtfully, or—in other words—to *think rationally.*

References

Irwin, Kathryn C. "Using Everyday Knowledge of Decimals to Enhance Understanding." *Journal for Research in Mathematics Education* 32 (July 2001): 399–420.

Lappan, Glenda, James T. Fey, William M. Fitzgerald, Susan N. Friel, and Elizabeth Difanis Phillips. *Bits & Pieces I: Understanding Rational Numbers.* Palo Alto, Calif.: Dale Seymour Publications, 1996.

Ma, Liping. *Knowing and Teaching Elementary Mathematics.* Mahwah, N.J.: Lawrence Erlbaum Associates, 1999.

Martinie, Sherri L., and Jennifer M. Bay-Williams. "Take Time for Action: Investigating Students' Conceptual Understanding of Decimal Fractions." *Mathematics Teaching in the Middle School,* in press.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000.

Schwartz, David M. *If You Hopped Like a Frog.* New York: Scholastic Press, 1999.

Silver, Edward A. "How Can Principles and Standards Help?" *Mathematics Teaching in the Middle School* 6 (September 2000): 20–23. ☐