Journal for Research in Mathematics Education
2012, Vol. 43, No. 4, 428–464

A Proposed Instructional Theory for Integer Addition and Subtraction

Michelle Stephan
University of North Carolina Charlotte

Didem Akyuz
Middle East Technical University, Turkey

This article presents the results of a 7th-grade classroom teaching experiment that supported students' understanding of integer addition and subtraction. The experiment was conducted to test and revise a hypothetical learning trajectory so as to propose a potential instructional theory for integer addition and subtraction. The instructional sequence, which was based on a financial context, was designed using the Realistic Mathematics Education theory. Additionally, an empty, vertical number line (VNL) is posited as a potentially viable model to support students' organizing their addition and subtraction strategies. Particular emphasis is placed on the mathematical practices that were established in this setting. These practices indicate that students can successfully draw on their experiences with assets, debts, and net worths to create meaning for integer addition and subtraction.

Key words: Conceptual knowledge; Instructional intervention; Integers; Middle grades, 5–8; Modeling

Historically speaking, the development of the integers was a complicated endeavor. Several articles detail the struggle that mathematicians had with integers, particularly with what it means to have numbers less than zero (Gallardo, 2002; Hefendehl-Hebeker, 1991). Attempts to devise instruction on integer concepts and operations have been no less troublesome. Integers are the first numbers that students encounter that require them to reason with numbers that cannot be modeled physically. Many studies have attempted to determine which model and real-world contexts would be most useful for supporting students' construction of integers.

The study reported in this article attempts to contribute to the debate about best models and contexts for teaching integers and extends the work done in this area by reporting the results of a classroom teaching experiment that took place over a 5-week period. In 2009, we engaged in a design experiment (see, e.g., Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) the aim of which was to study the implementation of an instructional sequence designed to support middle school students' development of integer concepts and the operations of addition and subtraction. To this end, the first author, who was a full-time middle school teacher implemented an instructional sequence on integers in her seventh-grade classes,

We would like to thank the teachers, administrators, and students at the middle school from which our data were collected. We would also like to thank the anonymous reviewers for their help in shaping the final version of this paper.

while the second author video recorded each class period. Our goal was to investigate students' use of a vertical number line (VNL) model as they engaged in instruction grounded in the context of assets, debts, and net worths. In this article, we report an analysis that was conducted on the whole-class discussions over the 5-week instructional period, and we document five classroom mathematical practices that emerged as students encountered situations in which they had to determine a person's net worth and transactions on it. The analysis is then used to suggest a proposed instructional theory for integer addition and subtraction.

RESEARCH ON INTEGERS

Historically, negative numbers were considered "absurd" early in their conception because mathematicians had not developed a way to understand numbers less than zero. Mathematicians questioned, in particular, why negating a negative number should result in a positive quantity. Students have similar difficulties, such as conceptualizing numbers less than zero; creating negative numbers as mathematical objects; and formalizing rules for integer arithmetic, particularly the meaning for the opposite of a negative number being a positive number. In the following sections, we (a) detail the conceptual difficulties that students encounter as they begin their exploration with negative numbers, (b) describe the contexts and models that researchers have explored for teaching integers, and (c) use this research to discuss the hypothetical learning trajectory that was implemented in a seventh-grade classroom.

Students' Conceptualization of Negative Numbers

Many researchers have interviewed, surveyed, and conducted teaching experiments (Steffe & Thompson, 2000) as a way to document students' conceptions of integers from very early ages (Bofferding, 2010) to early adolescence (Gallardo, 2002; Vlassis, 2004, 2008). By their very nature, negative numbers cannot be modeled with physical objects and have been labeled "fictive" (Glaeser, 1981). Five minus two, for example, can be modeled using five counters and removing two. However, when posed with the problem 2 – 5, young students struggle with taking 5 away when there are only two counters initially. Students often change the problem to 5 – 2 and subtract to get 3. Researchers argue that for a full comprehension of negative numbers, students need to be able to interpret the minus sign in multiple ways.

Drawing on Gallardo and Rojano's (1994) work, Vlassis (2008) argues that the negative sign can take on at least three meanings in mathematics: unary, binary, and symmetric functions. The *unary* operation identifies the quantity as a negative, in other words, the sign is "attached to the number" (Vlassis, 2008, p. 561), as in *negative* 10. A second function of the negative sign, and the most common way interpreted by elementary students (Bofferding, 2010), is *binary*. A negative sign functions as a binary operator when students interpret it as an action such as taking away, completing (as in how much more is needed to have 25, if you have 10), and

differences between numbers (Gallardo & Rojano, 1994). The third way to interpret the negative sign is as a *symmetric* function, where the symbol signifies taking the opposite of a number. For the expression – (–10), the first negative sign would signify the operation of taking the opposite of –10. Other researchers have described negative numbers similarly, as both a characteristic of an object and as a transformation (Thompson & Dreyfus, 1988) or a state versus an operator (Glaeser, 1981; Streefland, 1996). Based upon this work with students, researchers attempted to create models, grounded in a multitude of contexts, that would help students make sense of the multiple roles that a negative sign can play, as well as develop meaning for addition and subtraction of integers.

A Multitude of Models and Contexts: Which Is the Best?

As researchers sought to learn about the multiple functions of the negative sign, they also investigated the role that models and contexts could play in helping students understand them. Streefland (1996) and Peled, Mukhopadhyay, and Resnick (1989) examined whether operations with negative numbers could be modeled realistically or whether students would need to build their notions formally. Researchers have explored a variety of contexts (e.g., temperature, debts/assets) as well as models (e.g., number line) for teaching integers.

Contexts. In the last 20 years, much research has shown that grounding students' work in real-world contexts can be an instructional aid to the process of abstraction (De Lange, 1987; Gravemeijer, 1994; Stephan, Bowers, Cobb, & Gravemeijer, 2003). Other researchers have conducted experiments that seemingly contradict this premise and claim that students best transfer learning to abstract domains if they first encounter them in abstract ways rather than concrete contexts (see Kaminski, Sloutsky, & Heckler, 2008; De Bock, Deprez, Van Dooren, Roelens, & Verschaffel, 2011). Our intent is not to debate this issue; our work is based on the first idea, that learning in context provides students greater opportunities to abstract mathematical structure as well as meaning. Given the abstractness of the negative number, Janvier (1985) attempted to use helium-filled balloons and sandbags as a context, with balloons raising a basket and sandbags lowering it. Along the same lines, Davis (1967) developed a context around debts, with a postman who delivered checks and bills to a housewife who kept track of her money. To help students understand why the opposite of a negative number is positive, the scenario involved "misdelivering" bills and collecting them from the housewife. The positive and negative signs associated with checks and bills, respectively, would serve a unary function to describe the monetary value of each. Collecting and delivering letters would correspond with the more operational, binary function. Schwarz, Kohn, and Resnick (1993/1994) criticized both of these contexts as being too artificial for students. Mukhopadhyay, Resnick, and Schauble (1990) found that children can easily use assets and debts as a foundation for negative numbers. However, Schwarz, Kohn, and Resnick (1993/1994) criticized both the balloons and bills contexts as being too artificial for students. Schwarz et al. also criticized the assets

and debts context because it does not enable students to learn all aspects of integers, particularly because assets and debts cannot be thought of as directed magnitudes.

Among the other promising contexts that have been explored are positively and negatively charged particles (Battista, 1983), the activities of patrons in a disco (Linchevski & Williams, 1999), passengers on a bus (Streefland, 1996), lengths of positive and negative trains (Schwarz et al., 1993/1994), LOGO turtles moving along a horizontal number line (Thompson & Dreyfus, 1988), and a two-colored chips scenario (Lytle, 1994; Smith, 1995). Research shows that all these contexts have strengths, and when using them, students demonstrate a significantly better understanding of negative numbers. However, the contexts cited previously still had difficulty supporting students creating meaning for why the opposite of a negative "makes a positive." In addition, many researchers question whether students transfer their knowledge from the contextual world to other situations (Kaminski et al., 2008; De Bock et al., 2011) and so might question whether integers, given their abstract nature, should be grounded in abstract contexts instead. To address issues of transfer, Linchevski and Williams (1999) presented a study that used the design approach of Realistic Mathematics Education (RME) to create a disco scenario that they argued served as an authentic context for students. They designed their instruction to support students' modeling and symbolizing their actions with a double abacus. The instruction scaffolded students to symbolize their work in meaningful ways so that reasoning abstractly in other contexts was more viable. Although the design approach of RME is one way to address the challenge of transfer, the conditions under which students transfer knowledge from one domain to another are still elusive.

Models. Modeling in mathematics education has been a central focus for instructional designers. Designers search for a way to support students organizing their thinking that can be modeled/inscribed in the form of physical tools and symbols. The studies reviewed in the previous section posited different models for integer instruction that can be categorized mainly in two groups: *neutralization* and *number line* models, both of which attempt to build the unary, binary, and symmetry functions of the negative sign. Battista's (1983) electromagnetic charges context is an example of a neutralization model. To solve 5 – 8, Battista's model would have the student place 5 white chips (+ values) on a mat. Since there are not 8 positively charged chips available for subtraction, students would have to place down "zero-pairs" or 3 pairs of white and red chips, which sum to zero. Now there would be 8 white chips and 3 red, negative chips on the mat. To complete the operation, the student would remove the 8 positive chips and have 3 negatives remaining for a solution of –3. Other neutralization models used abacuses as concrete models for representing operations with integers (Dirks, 1984; Linchevski & Williams, 1999). Smith (1995) constructed a computer microworld called *Cy-bee Chips* as a neutralization model. The neutralization model draws heavily on cancellation of integers, or the unary function (Vlassis, 2008).

For the number line model Thompson and Dreyfus (1988) used a computer

microworld that simulated a turtle walking along a horizontal number line. Students were asked to input commands that would make the turtle travel and then asked to predict their position on the number line. Peled et al. (1989) argued that students intuitively draw on number lines to interpret operations with integers, in particular, creating a divided number line (two half-lines, a negative one and a positive one, put together) and a continuous number line. Students who reasoned with divided lines created special rules to cross zero and those who interpreted the number line continuously went up and down (adding and subtracting) fluently. The number line model attempts to support unary, binary, and symmetry functions of integers' signs, because –3 can signify a position, –3 can mean "move 3 units left," and –3 can represent the position opposite that of +3.

Very few studies have compared these two models but Liebeck (1990) showed that the students who used neutralization models performed slightly better on addition problems. However, both groups had difficulty in subtraction problems, especially ones that include different signs. Number line models have the advantage that ordering and symmetric understandings of negative numbers can be supported. However, linear models have a tendency to create artificial rules for the operations, especially why one moves a certain direction on the line for – (–).

An Emerging Instructional Theory for Integer Addition and Subtraction

Our goal in this project was to begin to develop a viable instructional theory for integer addition and subtraction. To this end, we designed a hypothetical learning trajectory, with conjectured learning goals, activities, model, discourse, and purposes; tested the trajectory in a classroom; analyzed the collective learning; and made revisions to the trajectory. The results of several, iterative experimentations in our classrooms supported a relatively stable *instructional theory* for addition and subtraction of integers. Taking into account the previously discussed literature, we decided to blend the context of finance with a number line model and created a hypothetical learning trajectory to be implemented with one seventh-grade classroom. Our rationale for this blending will be explained in the following sections as we detail the instructional design approach that guided our choices.

THE INSTRUCTIONAL DESIGN APPROACH

Realistic Mathematics Education

The approach that undergirds the design of the integers instructional sequence is Realistic Mathematics Education (RME). The roots of RME are based on the idea of mathematics as a human activity (Freudenthal, 1973). Freudenthal stated that people need to see mathematics not "as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics" (Freudenthal, 1968, p. 7). To mathematize, "one sees, organizes, and interprets the world through and with mathematical models. Like language, these models often begin simply as representations of situations, or problems, by learners"

(Fosnot & Dolk, 2005, p. 189). Students are guided to reinvent mathematical ideas through organizing realistic contexts that are didactically rich (Gravemeijer, 1994). In addition, students are encouraged to create and reason with models and mental imagery associated with the physical tools, inscriptions, and tasks they employ.

In terms of transfer of learning from one context to another, RME suggests that an instructional design should promote both vertical and horizontal mathematization. The instruction designed in this study attempted to help students mathematize vertically by staying within a context for an extended period of time, moving from concrete reasoning toward abstract reasoning with meaningful symbols. Because we had only 5 weeks for instruction, there was not enough time for the sequence to support horizontal mathematization, which would encourage students to reason about integers across contexts. However, we were able to assess students' abilities to use their knowledge in other contexts through interviews, final unit assessments, and pre- and postquestionnaires.

The Instructional Sequence

Simon used the term *hypothetical learning trajectory* (HLT) as a theoretical construct that refers to "the teacher's prediction as the path by which learning might proceed" (Simon, 1995, p. 135). An HLT has three main properties: learning goals, learning activities, and hypothetical learning processes. According to Simon's model, the teacher anticipates the types of mental activities in which students may engage during participation in the envisioned instructional activities and considers how the activities align with the teacher's goals. As an application of Simon's HLT, the RME theory can be used by instructional designers to create HLTs for longer term sequences of activities, not just 1 or 2 days (Gravemeijer & Stephan, 2002).

The HLT that served as the backbone for this study is organized in a table that is separated into five categories (Gravemeijer, Bowers, & Stephan, 2003): the tools, imagery, activity/taken-as-shared interest, possible topics of mathematical discourse, and possible gestures and metaphors (Rasmussen, Stephan, & Allen, 2004) that would support students' learning of integer operations. We have also broken the instructional sequence into six phases to delineate the proposed shifts in mathematical thinking that we attempt to support within the classroom.

According to Phase 1 of the HLT in Table 1, the first tool to be introduced to students is a net worth statement that lists the assets and debts that a person has. This net worth template builds on students' imagery of possessing assets and debts and the effect it has on one's financial situation. We expected that students would first interpret positive and negative signs (and the words that signify them) as having the unary function, negative and positive. We conjectured that students would connect with net worth, assets, and debts quite readily, especially at a time in which economic hardships abound around the world. We expected that students would construct the idea that a net worth is an abstract quantity, not a concrete entity that one can count like money, but rather the "status" of one's financial value.

Phase 2 of the HLT envisions students using net worth statements to solve problems in which they find net worths and compare them to each other. For example,

Table 1
A Hypothetical Learning Trajectory for the Integer Sequence

Phase	Tool	Imagery	Activity/ taken-as-shared interests	Possible topics of mathematical discourse	Possible gesturing and metaphors
1	Net worth statements	Assets and debts are quantities that have opposite effect on net worth.	Learning finance terms	• Conceptualizing an asset as something owned and a debt as something owed • Conceptualizing a net worth as an abstract quantity (not tangible)	
2	Net worth statements	Differences in collections of assets and collections of debts	• Determining a person's net worth • Who is worth more?	Different strategies for finding net worths	Pay off
3	Symbols (+ and –)	+ means asset and – means debt.	Determining and comparing net worths	• Different strategies for finding net worths • Creating additive inverses as objects	Pay off
4		Good decisions increase net worth. Bad decisions decrease net worth.	Which transactions have good and bad effects on net worth?	When taking away an asset; is this good or bad? When taking away a debt; is this good or bad? • Judging the results of transactions and, therefore, direction to move on a number line	Arms moving up and down to indicate good or bad movements

Table 1 (continued)
A Hypothetical Learning Trajectory for the Integer Sequence

Phase	Tool	Imagery	Activity/ taken-as-shared interests	Possible topics of mathematical discourse	Possible gesturing and metaphors
5	Vertical number line (VNL) *Model of* to *model for* transition	Empty number line to express (+ and –) movements	• Transactions • Reasoning with a number line to find a net worth after a transaction has occurred	• How do various transactions affect net worth? • Going through zero • The effect of different transactions • Different strategies for finding net worths	Arms moving up and down to indicate good or bad movements Pay off
6	Unknown transaction/ Net worth problems		Determining different possible transactions	Inventing integer rules + (+) = + −(−) = + + (−) = − −(+) = −	Pay off

students might be given a listing of the assets and debts of two people, and be asked to find and compare their net worths (all unary functions of positive and negative values). The conjectured taken-as-shared goal would be to determine which person is worth more than the other. We anticipated that students would create different strategies for finding net worths, such as (a) finding the *total* assets then *total* debts, and the *difference* between them; and (b) adding assets and debts one at a time until finished. We expected discussions about the "fictive" nature of a negative net worth (Glaeser, 1981), addressing the question of what it means for a person's net worth to be worth less than zero. Students would wrestle with the idea that most assets are tangible (e.g., a motorcycle) but debts are not (e.g., a mortgage). Even a net worth, whether positive or negative, is intangible and represents the difference between one's asset value and debt value. In the past, when we used only assets and debts, students could not reason through problems such as 5 – (–10) because the first number, 5, represented an asset and "you can't take away a debt of 10 when you don't have any debt, you only have an asset of $5." This prompted us to change the imagery associated with the number sentences. Imagine a situation in which a person's assets total $105 and debts total $100. The net worth would be $5 in this case. With the initial number in the sentence 5 – (–10) representing a $5 net worth (not debt), now it is possible to take away a $10 debt because there is a debt of $100 from which to take $10, leaving the net worth as $15. In other words, the structure and imagery underlying number sentences would now be the following: beginning net worth +/– (asset/debt) = new net worth.

Up to this point, the tasks used the words *asset* and *debt* to represent the unary nature of integers. Phase 3 of the HLT involves vertical mathematization, scaffolding students' symbolizing from reasoning with unsigned to signed integers. To this end, activities are posed that introduce assets and debts with the + and – signs rather than the words *asset* and *debt*. We expected students to continue to use their invented strategies for finding and comparing net worths. We designed some net worth statements intentionally so that a third strategy might emerge, that of cancelling assets and debts that are equal and working with the remaining assets and debts. This third strategy reflects the unary nature of the negative and positive signs in that +2000 and –2000 are each seen as positive and negative objects, respectively, that cancel each other's values.

Phase 4 of the HLT seeks to confront one of the conceptual stumbling blocks reported in the literature involving students making sense of the integer multiplication rules within a context. We developed the Good or Bad Transactions activities to help students build on their intuitions to construct an imagistic basis for the rules. Students were to judge the effect that various operations would have on a net worth. We would capitalize on students' gesturing as they described the increasing and decreasing effects on net worths. These tasks were written to have students create negative and positive signs as both binary and unary functions, in which the first sign signifies an action/transformation (Thompson & Dreyfus, 1988) and the second represents the status of the quantity (negative/debt or positive/asset). Thus, – (–100) would be read as "taking away a debt of $100" and – (+90) would mean

taking away an asset of $90. Students were asked to judge whether the symbols signified what they called a *good decision*, one that makes the net worth get better or a *bad decision*, one that decreases the net worth.[1]

Phase 5 of the HLT incorporated tasks that we hoped would encourage students to find the effect that various transactions had on a person's net worth. We called these *transaction tasks* because they involved taking a person's original net worth, performing a transaction on it (such as adding a debt), and determining the new net worth. Although some students would perform erroneous calculations, we expected some students to "go through zero" as predicted by Peled et al.'s (1989) research. Inspired by previous work with elementary students' mathematical reasoning with a horizontal number line (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997), we decided that the teacher could introduce an empty VNL to make students' going-through-zero strategy more visual for the other students. In this way, the VNL would emerge in the class from student thinking as a *model of* transactions on net worths. Later, it might evolve to become a *model for* the formal integer operations of addition and subtraction (Gravemeijer & Stephan, 2002) in less context-dependent problem situations.[2]

In Phase 6 of the HLT we designed activities that asked students to determine the results of various transactions, including multiple ones such as 100 – 2 (–50). At first, the problems were posed in context and moved toward number sentences. To formalize the rules for integer operations, students were asked to list various transactions that could have taken place to make Chad's net worth change from $10,000 to $12,000. We thought that students might construct a variety of number sentences by changing the unknown transaction, for example,

$$10{,}000 + \mathbf{(+2000)} = 12{,}000,$$
$$10{,}000 - \mathbf{(-2000)} = 12{,}000,$$

or others such as

$$10{,}000 - \mathbf{(-1000)} - \mathbf{(-1000)} = 12{,}000.$$

THE CLASSROOM CONTEXT

The integers instructional sequence was implemented in a public middle school (grades 6–8, ages 11–13) in Central Florida, United States. There were approximately

[1] We are thankful to reviewers for pointing out the ethics of allowing students to think that when a person incurs a debt that this is always a bad decision. Subsequent implementations of this instructional sequence will involve discussions about the value of incurring debts.

[2] We have written about the tension between teacher-initiated and student-created symbolizations elsewhere (Stephan, Cobb, Gravemeijer, & Estes, 2001). Is it better for students to introduce their own—often informal—symbolizations, or should the teacher "impose" his or hers? We resolve this tension by suggesting that in classroom situations, the teacher should introduce tools that fit with students' current reasoning. In this way, teaching can progress in an efficient manner and the tool can emerge in a bottom-up manner that fits with students' reasoning rather than serving as a "teacher's" tool that must be "decoded" by students.

1500 students enrolled in the school, which served students from a variety of backgrounds, primarily middle class. When the experiment was conducted, the first author had 3 years full-time teaching experience. Prior to that, she had been a mathematics education professor and specialized in designing instruction to support standards-based teaching approaches consistent with *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000). The class consisted of 13 boys and 7 girls. Three of the boys had been identified as having learning disabilities and over half the class was performing below grade level. There was also a 10-year veteran coteacher who was certified in special education. Stephan and her coteacher had cotaught for 3 years.

The classroom teaching experiment was conducted in the third quarter of the school year. Therefore, norms consistent with establishing a standards-based environment (consistent with NCTM, 2000) had already been set and were relatively stable during the instruction reported here (Akyuz, 2010). Generally, the teacher's roles included introducing the task, letting students explore individually or with partners, and facilitating whole-group discussion afterwards. During facilitation time, her main obligations were (a) to manage turn-taking by students so that everyone had access to the discussion; (b) to repeat or to clarify a student's contribution; (c) to ask for (dis)agreement; (d) to choose students to present so that mathematically significant discussions could occur; and (e) when appropriate, to introduce symbolizations to help students organize their reasoning. Although all teachers embody authority in the classroom, she attempted to establish norms such that students should disagree with others, including her, whenever necessary. Space does not allow us to show evidence of these norms, but Akyuz's (2010) analysis showed that students in this class routinely indicated (dis)agreement, sometimes prompted by the teacher and other times without prompting.

The teacher and researcher conducted pre- and postinterviews with each student to assess their understanding of integers prior to and subsequent to instruction. In addition, pretests and posttests were administered to each student to document learning from a quantitative point of view. All pretest and posttest questions were drawn from Smith's (1995) study and included both procedural and conceptual problems. Both preinterview and pretests indicated that students had some previous knowledge of integer operations, and students were far more successful with addition of integers than with subtraction of integers. Most students were able to solve addition problems successfully. However, their difficulty with 5 – (–7) and –4 – (+8) revealed that, for the majority of our students, the minus sign between the two numbers (displayed to students as the same-size sign as that used for indicating a negative number or opposite of a number) signified the act of subtracting two quantities (7 – 5 or 8 – 4) and then attaching the sign of the greater quantity to the answer. Additionally, when justifying both correct and incorrect answers, it was clear that students had a superficial understanding of the operations and rules that they often erroneously employed.

Once the integer instruction began (in March, 2009), data collection continued for 5 weeks, with the class observations recorded through video and audio recordings,

field notes taken by the second author, daily interviews of the teacher (first author) by the second author, and the weekly meetings consisting of three other seventh-grade teachers and both authors. At the time of the experiment, the school district had adopted *Connected Mathematics Project* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998), a reform textbook, and advocated a student-centered approach to teaching. Units within this textbook take at minimum 4 weeks to teach; thus, 5 weeks was a realistic and acceptable time period for our instruction.

METHOD

The theory on which we draw to make sense of students' learning is a version of social constructivism called the *emergent perspective*. This perspective has been described more specifically elsewhere (e.g., Cobb & Yackel (1996); Stephan, 2003). Briefly, this theory draws from (a) constructivist theories that specify learning as an organic, auto-regulated series of cognitive reorganizations (e.g., Steffe, von Glasersfeld, Richards, & Cobb, 1983; von Glasersfeld, 1995) and (b) interactionist theories that emphasize learning as a social accomplishment (e.g., Bauersfeld, 1992; Blumer, 1969). From this perspective, students are viewed as reorganizing their learning as they both participate in and contribute to the social (and mathematical) context of which they are a part.

According to Cobb (2003), one criterion related to conducting classroom-based research is that analyses should feed back to inform the instructional theory. In Cobb and his colleagues' work, they suggest that one way to assess the viability of an instructional sequence is to document both the classroom mathematical practices and individuals' ways of participating in and contributing to them (Cobb, 2003; Stephan, Bowers, Cobb, & Gravemeijer, 2003). In this article, we fulfill one of these criteria by documenting the learning of the community in the form of classroom mathematical practices that were established as we implemented the HLT described previously. Space limits us from presenting complementary case studies of individual students' reasoning. Nevertheless, the practices that become taken-as-shared by the classroom are strong indicators of the collective learning that developed as the instruction was implemented. Stephan et al. (2003) suggest that the classroom mathematical practices give a picture of the *actualized learning trajectory*, the way in which the HLT became constituted in practice. The mathematical practice analysis will provide necessary feedback to make changes to the emerging instructional theory on integers.

Analysis Method

The method used to analyze the collective classroom mathematical practices was constructed and described in Rasmussen and Stephan (2008) and in Stephan and Rasmussen (2002). This qualitative method uses Krummheuer's (1995) adaptation of Toulmin's (1958) model of argumentation to analyze the collective discourse over the course of several weeks of instruction. In his seminal work, Toulmin

created a model to describe the structure and function of certain parts of an individual's argument. Krummheuer (1995) adapted Toulmin's constructs to analyze students' *mathematical* arguments. Figure 1 illustrates that an argument consists of three parts: the *data, claim*, and *warrant*, with the warrant often left implied rather than spoken. Individuals may give aid in the form of a *backing* to support the warrant of their claim in addition to *rebuttals* and *qualifiers*. In argumentation, the speaker may make a claim and present evidence or data to support that claim. The data typically consist of facts or procedures that lead to the conclusion that is made. For example, imagine that students were asked to solve the problem 25 – (–10). During a discussion, Jason, a fictitious student, makes a claim that the answer is 35. When prompted by the teacher to say more about how he got his answer, Jason says, "I just added 10 to 25." In terms of Toulmin's constructs, Jason has made a *claim* of 35 and given evidence (*data*) in the form of his calculation for obtaining his answer. Jason might also include a *warrant*, or license, to clarify how his evidence relates to the conclusion. Oftentimes, the content of a warrant is algorithmic (Forman, Larreamendy-Joerns, Stein, & Brown, 1998) in that the presenter states more precisely the procedures that led him to the claim. In our example, Jason might provide the following warrant: "I started with a net worth of $25, and taking away a debt adds $10, so the new net worth is $35."

Students may now see how Jason went from data to claim with such an explanation, yet not understand or agree with the content of the warrant used: "I see that you added to get 35, but it says take away a debt, so why are you adding?" The mathematical authority (or validity) of the warrant is being challenged and the presenter must provide a backing to justify why the warrant is valid. Jason might then provide a backing by stating, "We already decided that when you take away a debt, that is a good thing. It makes your net worth go up. So, if you were worth $25 and your debt of $10 was forgiven, now you are up to $35." Jason has validated his actions of adding when there are two subtraction signs. In this case, Jason's claim and subsequent argumentation were mathematically correct, but there are instances in which parts of the claim may be invalid and he would need to qualify or withdraw his claim.

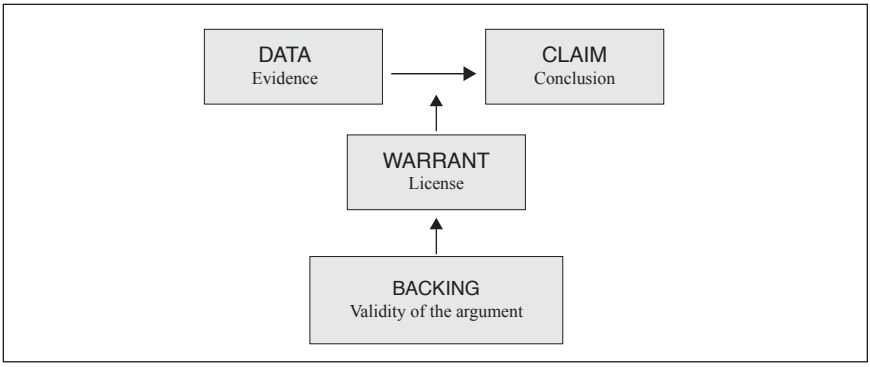

Figure 1. Toulmin's (1958) model of argumentation.

Toulmin (1958) detailed the structure and function of an individual's argumentation, but analyzing a classroom argumentation that involves multiple speakers is more difficult. To document the classroom mathematical practices, we used the method described in Stephan and Rasmussen (2002). To analyze classroom argumentation, we made transcripts of all whole-class discussions. We then conducted a three-phase analysis in which we coded all thematic discussions using data, claims, warrants, and backings. If student challenges were made during the episode, they were noted as part of the coding along with the final claim (possibly a qualified claim) that became taken-as-shared. After Phase 1, we had created a 44-page argumentation log that summarized the structure and function of students' contributions in whole-class discussions. Phase 2 then took the argumentation log as data to determine (a) when students challenge a claim; (b) when a previously challenged claim no longer needs justification, that is, when either backings or warrants dropped out of students' arguments; and (c) when the statements made in claims shifted position in students' arguments and were used as data in subsequent arguments. According to Stephan and Rasmussen (2002), such occurrences signal that a mathematical idea has become taken-as-shared. We then made a list of all the mathematical taken-as-shared ideas.

Finally, in Phase 3, Stephan and Rasmussen (2002) take the list of taken-as-shared mathematical ideas established by the classroom community and organize them into mathematical practices. Classroom mathematical practices are the researcher's way of organizing the content-specific, student-generated mathematical ideas that are taken-as-shared. In our experiment, five mathematical practices emerged (see Table 2).

Each of the five mathematical practices lists the two to four mathematical ideas that constituted them. It is important to note that an analysis of mathematical practices is reliable only if social norms of explaining, indicating (dis)agreement, and asking questions when needed are taken-as-shared. Imagine a classroom in which students do not explain, but only give their answers. Our analytic technique would not be useful because there would probably be few warrants or backings and little data to be analyzed. There would be no evidence of challenges that illustrate subsequent negotiations that typically result in taken-as-shared mathematics. In addition, given the inherent authority of the teacher in discussions, if the students do not feel free to indicate disagreement or explain, then we would not be able to trust that the explanations students are giving are legitimately theirs rather than the thoughts of the teacher. We claim that the forthcoming analysis is reliable, because Akyuz (2010) showed that the social norms important for reliable analysis were taken-as-shared.

CLASSROOM MATHEMATICAL PRACTICES

Five classroom mathematical practices were documented over the course of 19 class periods. In the paragraphs below, we describe the mathematical ideas that became taken-as-shared as students established these practices. At times we provide an analysis of the discourse using the constructs *data*, *claim*, *warrant*, and *backing*, but to do so for every mathematical idea would be overwhelming. We therefore

Table 2
Five Classroom Mathematical Practices

Classroom Mathematical Practices
Practice 1: Interpreting net worth as a positive or negative quantity • Net worth is a combination of a positive and a negative value [when the assets and debts are both nonzero]. • When a negative value is greater than [in absolute value] a positive, the combination is negative.
Practice 2: Using zero as a point of reference for calculations • Referencing zero to determine net worth • Referencing zero to compare two net worths • Referencing zero to add or subtract integers • Cancelling equal positive and negative quantities
Practice 3: Comparing integers using a vertical number line • Higher [in absolute value] negative numbers are farther away from zero. • Structuring the gap between two integers to find the difference
Practice 4: Reasoning with a vertical number line to determine the results of addition and subtraction operations • Transactions can have a positive or negative effect on a quantity. • A vertical number line can be used to find the results of integer operations. • Subtraction of integers is not commutative.
Practice 5: Determining the meaning of positive/negative signs • Different operations (transactions) can have the same effect on a quantity. • A minus sign is different from a negative sign.

analyze the argumentation sparingly to give the reader a sense of how we used argumentation to determine when an idea was taken-as-shared.

Practice 1: Interpreting Net Worth as a Positive or Negative Quantity

Two mathematical ideas became taken-as-shared as students mathematized the financial situations that were posed to them:

- Net worth is a combination of a positive and a negative value [when the assets and debts were both positive]; and
- When a negative value is "greater than" [in absolute value] a positive, the combination is negative.

The instructional sequence began with an experientially real context of determining a person's financial net worth. The teacher asked students if they had ever heard of Oprah Winfrey, a famous celebrity in the United States.

> I Googled Oprah this morning before you came to class and found out that her net worth is 1.5 billion dollars. Can you imagine that? How many zeroes is that? What does that mean to say that she is worth 1.5 billion dollars?

Students suggested that this means she has a lot of cash in the bank, she owns yachts, buildings, production studios, and so on. They did not mention that she probably owes money, so the teacher asked whether they thought that Oprah had any debt. Students argued that she probably has a mortgage on her million-dollar mansions, loans from the bank to finance her studio, and other examples. The teacher kept the student-generated list on the board with assets in one column and debts in another column, then introduced the terms officially: *Assets* are what you own, and *debts* are what you owe.

The teacher then told students that she had recently hired a financial advisor to help her save money so that she would have enough to send her child to college. Her financial advisor sent her the following net worth statement and asked her to complete it (see Figure 2).

Once students had a copy, they were encouraged to examine this page to find familiar and unfamiliar words. After discussing unknown terms, the launch of the instructional sequence (Phase 1) ended when the teacher announced that the class was going to be using financial worth statements similar to this to determine a person's net worth.

Net worth is a combination of a positive and a negative value [*when the assets and debts were both nonzero*]. To make meaning for operations with integers, students must first understand how assets and debts work together to form a person's net worth. On Day 2 of instruction, students were given Brad's and Angelina's fictitious net worth statements and asked to find their net worths (see Figure 3). Based on the statement, Angelina's net worth is +$90,000 and, other than arithmetical errors, students had no difficulty calculating this net worth. Below is a conversation in which Charlie defended his claim of $90,000:

T: Let's talk about this. For somebody like Betty who was not here yesterday, who can explain how we get that [90,000]?

Charlie: All you do is you get 940,000 [total assets] and subtract it from [*sic*] 850,000 [total debts].

T: Why? Because they [a couple of students] were not here yesterday. They are like . . . what is that debt all about? Can you explain?

Charlie: Debt means you owe 850,000 and her total asset is how much she has.

T: This is how much she owes and this is how much she owns [records that on the board]. She owes 200,000 in boat loans and then she owes a penalty for pulling out of a movie deal. So you say she is in debt that much altogether [850,000]. And for assets you say it is what she owns, that is how much she's worth. You said you find the difference between those two. She is worth 90,000 altogether once you take out her debts. All right, what about this other one?

Net Worth Statement

Client Name

Cash Assets	**Current Value**
Cash Bank Accounts	
Money Market Accounts	
Other Cash	
Investments	
Bonds	
Stocks	
Mutual Funds	
Annuities	
IRAs	
Retirement Plans	
Real Estate	
Other Investments	
Personal Assets	
Household Contents	
Primary Residence	
Automobiles	
Other	
Total Assets	
Debts	
Mortgages	
Personal or Business Loans	
Automobile Loans	
Credit Cards/Charge Accounts	
Other Debts	
Total Debts	
Net Worth	

Figure 2. An empty Net Worth Statement template.

Net Worth Statement

Client Name: *Angelina*

Cash Assets	**Current Value**
Checking account	
Money market accounts	
Savings account	$100,000
Investments	
Restaurant	$500,000
Owns a movie production company	$250,000
Owns land in Namibia	$90,000
Other	
Personal Assets	
Total Assets	
Debts	
Boat loan	$200,000
Penalty for pulling out of a movie deal	$650,000
Total Debts	
Net Worth	

Net Worth Statement

Client Name: *Brad*

Cash Assets	**Current Value**
Cash bank accounts	$150,000
Money market accounts	
Savings account	
Investments	
Owns a Planet Holywood	$450,000
Mutual funds	
Real estate	
Other	
Personal Assets	
Car	
Other	
Total Assets	
Debts	
Owes George Clooney gambling debts	$ 90,000
Auto loans	$175,000
Owes Jennifer Anniston a divorce settlement	$525,000
Total Debts	
Net Worth	

Who is worth more money when Brad and Angelina get married? Explain in complete sentences.

Figure 3. The first activity for finding the net worth of two people.

In this exchange, Charlie made the claim that Angelina's net worth is $90,000. Upon the teacher's request, Charlie provided the data for this claim by stating that he found the total assets and the total debts and subtracted the two. Aware that some students were absent the day the sequence was launched, the teacher took this opportunity to call for further justification by asking why Charlie performed those calculations and what those numbers represented. Charlie attempted to provide a warrant when he said that "debt means you owe 850,000" and assets are the "money you have." Although Charlie is using *owe* and *own* to describe positive and negative attributes, we see this as a linguistic stepping-stone to building + and – unary symbols. In fact, other students had already begun to use the + and – signs to symbolize net worths.

Because Charlie was a student with a learning disability and had difficulty expressing his thoughts in language, the teacher, as was her common practice,

elaborated his warrant, in this case for the benefit of students who had been absent for the launch of the sequence the previous day. Together, she and Charlie created the warrant for this argument: To find a net worth, you find the difference between total assets and total debts.

This is the second instance in which finding the difference between total assets and total debts arose in students' conversations. Later in that class period, the idea of finding the difference between assets and debts became a topic of conversation again as students defended –$190,000 as Brad's net worth. The backing that was provided by Danny was that Brad is in debt because he "already paid some of them off [debts], but he did not pay all of it, so he still has 190,000 left over." On the third day of the instructional sequence, evidence from students' arguments indicated that net worth as a combination of a positive and negative value had become taken-as-shared during these conversations on Day 2. Students used this idea as data in their arguments on Day 3 without the need for backings, which indicates it had become taken-as-shared.

When a negative value is greater than [*in absolute value*] *a positive, the combination is negative.* The Brad and Angelina task was intentionally designed so that Angelina's total assets were greater than her total debts and Brad's debts were greater than his assets. We designed it this way so that students would be confronted with the idea of "going below zero," the same idea that bothered the mathematics community initially.

Students engaged in a discussion that focused mainly on the proper way to calculate Brad's net worth. Students drew on their algorithms for subtracting whole numbers, with some students placing the total assets on top of debts and others vice versa. Students who set up the problem as in Figure 4(a) simply subtracted from the bottom number upward (i.e., 9 – 0 = 9 and 7 – 6 = 1) to get 190,000 and then took the "–" sign from the bigger number. Students who solved it as in Figure 4(b) argued that the bigger number always goes on top, subtracted the traditional way to get 190,000 and that, as Bradley put it, "his debt outweighs his assets, he is still in debt some money," so it would take a negative sign. Argumentation analyses show that this idea became taken-as-shared during this conversation as students no longer had to justify putting a negative sign on a net worth when the debts outweighed the assets. In fact, students used this idea on Day 3 as backing for another conclusion that shows that by Day 3 "when a negative value is greater than a positive, the combination is negative" was taken-as-shared. Because the intent of the instructional design was to help students operate on integers with the aid of a VNL, the teacher did not focus much on the errant algorithms students invented for finding the difference between total assets and total debts. Subsequent instructional activities would target students' algorithms for calculating the difference.

Practice 2: Using Zero as a Point of Reference for Calculations

The idea that using zero as a point of reference was very intuitive for students and arose as warrants, backings, and data for many arguments throughout the entire

600,000	790,000
– 790,000	– 600,000
(a)	(b)

Figure 4. Students' attempts to find net worths when debts outweigh assets.

sequence. We identified using zero as a reference as a practice that spanned the entire instructional period and therefore is associated with four taken-as-shared ideas:

- Referencing zero to determine net worth
- Referencing zero to compare two net worths
- Referencing zero to add or subtract integers
- Cancelling equal positive and negative quantities

Referencing zero to determine net worth. The first instance in which zero was used as a reference occurred on Day 2 as students were struggling to make sense of Brad's negative net worth. As students attempted to justify why Brad's net worth was negative, Danny introduced the metaphor of "paying off" for the first time when he argued that Brad had "already paid some of them off [debts] but he did not pay all of it so he still has 190,000 left over." In other words, Danny defended his answer of –$190,000 by stating that he imagined Brad paying off as much debt as he could (i.e., bringing his total assets to zero) and still having a negative balance. By Day 3, other students used pay off as backing for their strategy of finding the net worth of a person whose net worth statement appears in Figure 5.

As students struggled with their algorithms again for finding this person's net worth, the teacher used Nathan's pay-off argument as an opportunity to introduce a VNL that was used by a student in a different class:

Tisha: I just added all the negatives, and I got negative 8400, and 8400 minus 8000 positive and I got 400, negative 400 [writes –8400 + 8000 in vertical format].

Flora: I did minus [subtract] 8000 from 8400 [8400 – 8000], and she did the same thing and got 400.

T: How did you know to put negative by it? How did she know that she would write negative, Charlie?

Charlie: Because she owes more than she has.

T: [To Nathan] Can you re-explain it?

Nathan: Yes. Since he was negative, 8400 is negative, so she subtracted by the positive 8000. Since it is more than zero, he is still in the negative.

T: What does it mean? Marsha, what does it mean his net worth [is –400]?

Marsha: If he uses all his assets to pay off his debts, that is how much he would need more.

T: Stuart, what did she just say?

Stuart: He pays off debts by using assets.

	–$2900
	–$3700
	–$1800
	+$8000
Net Worth:	$

Figure 5. An individual's net worth statement.

T: So he uses his assets to pay off his debts?
Stuart: Yes.
T: Can you tell them what happens? Bradley, finish that.
Bradley: He actually has assets, then his debts overwhelmed his assets.
Norman: When he *pays off* debts from his assets that is how much he had left.

In this exchange, multiple students used the metaphorical imagery of *paying off* to justify why 8400 is decremented by 8000 and that the result must be a negative number. From a more mathematical point of view, –8400 was being partitioned into –8000 and the remaining (–400), and –8000 was cancelled (paid off) leaving the remaining –400. In this case, the value of the assets was insufficient for "paying off all debts," so the leftover money was negative. Because students had agreed that the net worth was –400, the teacher took this opportunity to introduce a number line that was used by Dusty during preinterviews, as well as by Sienna, a student from a different class:

T: Do you know Sienna in my second period class? She had this really cool way. If you had 8000, she modeled it on a number line, an up-and-down vertical number line. He has 8000 assets and pays it off as much as he can. What would that take him to?
Dusty: Zero.
T: Do you agree with that Betty? What about you, Tisha? Do you agree with that?
Tisha: [Yes] That when he pays it off, he will go down to zero.
T: Can he pay off all his debts?
Students: No.
T: Since you say no, what is left [of the total debts of 8400]?
Seth: Negative 400.
T: So he goes to negative 400. She [Sienna] called this a pay off too. That it's gonna take 8000 dollars to pay off and then he is still in debt [models this on the number line; see Figure 6].

The notion of referencing zero as a way to determine the net worth emerged from students' intuition, but students were struggling in this excerpt with ways to symbolize the results of paying off. Prior to using the VNL, students were struggling

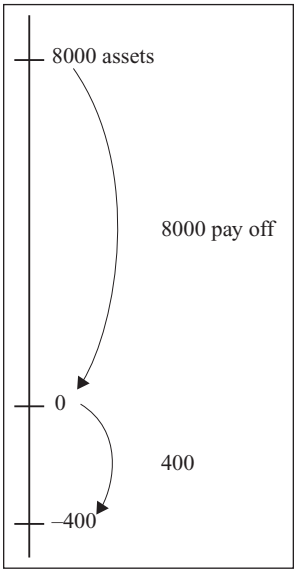

Figure 6. Sienna's number line model.

with whether to put –8400 on top of +8000 in the traditional format learned from elementary school or to put 8400 – 8000 devoid of signs and adding the sign after finishing the calculation. Similar to Murray's (1985) findings, one of our students used a number line during the preinterviews, and another student from a different class spontaneously reasoned with a number line in the teacher's class that met until 10 minutes before this one. Inspired by Dusty and Sienna's strategies, the teacher introduced the VNL to support students' imagery of partitioning an integer around zero as a reference point. In this way, the VNL was introduced in a bottom-up way to help students organize their partitioning strategy in a more visual way.

On Day 4, the teacher posed a problem as a quick assessment: Cody, one of the students from class, has total assets worth 325 and total debts worth 450. What is his net worth?

T: All right, are you ready? Marsha, Sally, are you ready? Let's start by getting some different answers up there first. We may have nothing to talk about. Danny, what did you guys get?

Danny: Being in debt 125.

Tisha: Negative 125.

Student: 75.

T: Who do you want to hear about first?

Stuart: They are all wrong.

T: What is your answer?

Stuart: 135.

T: [Teacher starts with Danny] If you got something different, you know your job. Stuart, you know your job, especially if you do not agree with Danny.

Danny: He has some debts, and the debts are more than assets. He will get to zero first and then down to 125. Because it is opposite to do (he writes 450 – 325 vertically). He has this much, and he has debt this much. I subtracted it so I am not sure how I got it.

Danny argued that Cody "has some debts, and the debts are more than his assets. He will get to zero first and then down to 125." Danny referenced zero as data for his argument and no other student contested it, which indicates that partitioning an integer around zero to find a person's net worth was taken-as-shared at this point. Students who had different answers realized that they had made subtraction errors and readily accepted Danny's strategy. Danny struggled with verbalizing why he

subtracted the two amounts, but as we will see subsequently, the number line began to facilitate students' explanations.

Referencing zero to compare two net worths. Although referencing zero emerged as students determined a person's net worth, it again arose during tasks in which students were asked to find out how much more one person was worth than another. On Day 5 we posed a problem on a VNL split into two colors, the top half black and the bottom half red. The teacher related the number line to accountants' use of the terms indicating that when someone's net worth is negative, it is said that he or she is "in the red"; when someone's net worth is positive, he or she is "in the black." The task involved finding how much more Gilligan was worth than Mary Ann (see Figure 7):

Sally: 5000.

Dusty: 1000.

Gage: She [Mary Ann] wants to get as much as Gilligan, but she still owes money from that car. So to get as much money as him, 3000 dollars, she would have to pay off her car debt, basically, to get to zero.

T: So which one [which answer] can help you with your argument about this?

Gage: This one [5000]. Because after she pays off her debt [draws an arrow from –2000 to 0], she is back at zero and she has no money. But after she worked in her three jobs, she got her paycheck and she got 3000 from her paycheck. So after that, she had to get a total of 5000 dollars to get up to how much he had.

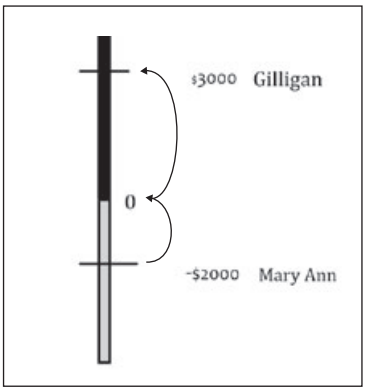

Figure 7. Gage's activity with a net worth number line.

In terms of Toulmin's (1958) argumentation scheme, we interpret Gage's statements as making a claim of $5000. For his data, he imagined hypothetical situations in which Mary Ann would make enough money to pay off her debts to get to zero (no money) and then earn more money to get her equal with Gilligan. His warrant contained his actions and recordings with the VNL to show how his data linked to 5000. In this instance, referencing zero to compare two net worths was introduced in the data of a student's argument and was beyond validation, that is, no backing was needed. No other students challenged Gage's argument, and we concluded that referencing zero to compare net worths was taken-as-shared at this point in instruction. Argumentations from future classroom discussions substantiated our conclusion. It is worth noting here that Gage used the visual characteristics of the VNL to illustrate his reasoning; therefore, the concreteness of students' reasoning on the empty number line was critical to the development of this practice.

Referencing zero to add or subtract integers. We introduced transaction problems for the first time on Day 6 of instruction. On Day 7 students were presented with Kim's net worth statement, which showed a variety of assets and debts in random order. Students were asked to determine her net worth [–$500] and then to consider how that net worth would change if Kim's dad took away (forgave) Kim's loan of $3000. Some students concluded that her new net worth would be 2500 and others 3500. Students who answered 3500 merely added the two amounts together, and it was difficult to convince them that they were wrong. The teacher asked students who claimed Kim's net worth was 2500 to help her show it on a number line.

T: Where did she start off? [Draws a vertical number line as in Figure 8]

Bradley: –500.

T: And this thing happens to her is a good thing or bad thing?

Students: Good.

T: Which direction are we going to go, Adam?

Adam: Up.

T: Betty. How much are we going to go up? How much of a good thing was this to her?

Students: 3000.

T: The good thing was $3000. Where would she end up?

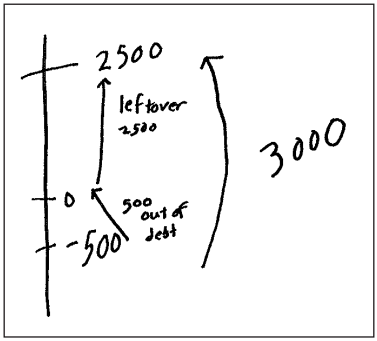

Figure 8. Collectively constructed number line.

Students: Positive.

Stuart: You take 500 to get to zero. So, then you automatically take 3000 minus 500. [T asks Stuart to record this on the number line.]

T: Why is he doing this part? Danny, why is he doing that part [3000 – 500]?

Danny: It will take him up to zero. Because it will take you to what number you land on.

Flora: She paid off 500.

T: So, she paid off 500. So, now what?

Flora: Then you pay off 2500 to get 3000 altogether.[3]

In this episode, we see the students and the teacher struggling to determine the results of a transaction in which Kim's net worth changes for the better. To help students interpret the results of their actions, the teacher drew a number line and asked students to record their calculations on it. The teacher called for a warrant for what led Stuart to subtract 500 from 3000, and Danny and Flora repeated that

[3] In hindsight, we wish that the teacher's facilitation here would have been less directed. In the moment, the teacher believed that all students knew that –500 would be placed below zero and that taking away a debt of 3000 would result in going up. Looking back, we wish that the teacher had let students model it on the number line themselves, rather than funnel the beginning portion of the argumentation. Regardless of the directness of the initial part, she allowed other students to guide the class regarding how much the new net worth would be.

Stuart was paying off 500 and had 2500 left to make 3000. Using zero as a reference point did not require a backing (why this is a valid strategy) from students, and we concluded that referencing zero to add/subtract integers was taken-as-shared at this point. The teacher asked Cody to use the number line to defend his incorrect answer of $3500. When he came to the board and reasoned through it, he could not justify 3500 with the number line and actually proved that the result should be 2500.

To solve transaction problems, students routinely drew a number line and used the metaphorical imagery of *paying off* to justify why they added or subtracted to obtain the new net worth. The argumentations in this classroom indicate that using zero as a reference point was intuitive to students and emerged from them on the second day of instruction. *Pay off* was an imagery on which students drew to solve a variety of problems, including finding a person's net worth, finding the difference between two or more net worths, and making sense of adding or subtracting two integers. Students moved quickly to solving a variety of context-free number problems such as 2000 + (–1000), 2000 – (–1000), and –2000 – (+1000). Students were observed either drawing a number line and going through zero or gesturing in such a way that indicated they were reasoning with the imagery of a number line.

Cancelling equal positive and negative quantities. A final mathematical idea that became taken-as-shared in the collective discourse involved cancelling out positive and negative numbers that were equal in absolute value (i.e., they were additive inverses). This method first arose during an activity on Day 3 but became established during discussions of the *American Idol* (a popular television singing competition) problem on Day 4. Students were given fictional net worth statements for Paula, Randy, and Simon, and we created the numbers in such a way that the cancellation method might arise from students. In particular, Randy's net worth statement contained the following assets and debts: +$1000, –$1000, +$2000, +$500, –$2000, –$500, and +$60. Although many students found Randy's net worth by totaling assets, totaling debts, and finding the difference, Seth cancelled out integer opposites. Students referred to this as *Seth's Cancellation Method* and the words *additive inverse* were posted on the word wall to signify integers that summed to zero. In ensuing activities, students referred to Seth's method as data or warrants for their calculations, which indicates that reasoning with additive inverses had become taken-as-shared.

Practice 3: Comparing Integers Using a VNL

Mathematical Practice 3 that emerged as students engaged in the instructional sequence involved comparing integers in the form of net worths using a VNL. Two mathematical ideas became taken-as-shared as students solved problems such as this: Paris' net worth is –$20,000, and Nicole's net worth is –$22,000. Who is worth more and by how much?

- Higher [in absolute value] negative numbers are farther away from zero
- Structuring the gap between two integers to find the difference

Higher negative numbers are farther away from zero. On Day 5 students were given several problems in which they had to compare celebrities' net worths. During discussions of the Paris and Nicole problem, one student argued that Nicole is worth more, and placed her amount of –22,000 above –20,000 on the number line. This was immediately rejected by students:

Adam: 20,000 is supposed to be before [higher than] 22,000.
T: Did you hear that Charlie? Do you agree with that? [Nods agreement] Does it matter guys? Why?
Adam: Paris is closer to zero.
Charlie: Because Nicole owes more, so she has to be in the red more.
T: She has to be more in the red. Nathan? He says because Nicole owes more. How do you know she owes more?
Charlie: It has already said it. Negative 22,000 and negative 20,000.
T: Dusty, what do you want to say?
Dusty: It should be opposite of going up to zero.
Mark: I think we should put the less number in front of the higher number.
T: In front of it, like this? [Puts 20,000 above 22,000 on the vertical number line]. Why?
Mark: Because 20,000 is closer to 0.
T: You guys keep saying that. What do you mean? Marsha?
Marsha: Yes, there is a reflection, if you, like, flip it [the top half of the number line] upside down.
Charlie: Because –20,000 is being closer to out of debt than –22,000.
Bradley: The reason 22,000 should be farther down is because it is further down in the hole. Like you owe more than the other person.

In this brief exchange, students immediately rejected Nathan's number line that showed –20,000 lower than –22,000. Their backings to support the correct placement drew on a variety of imageries, a reflection line, being closer to zero, being deeper in the hole, and being closer to out of debt. The number line model, similar to that proposed by Thompson and Dreyfus (1988), enabled students to discuss the ordering of positive and negative numbers, a deficit of the neutralization models. This was the only time that ordering negative numbers emerged as a topic of conversation and that students correctly ordered them in subsequent argumentations. Students took it for granted that "higher" negative numbers would be farther down away from zero.

Structuring the gap between two integers to find the difference. As students solved problems that asked them to compare two net worths, structuring the gap between two integers became taken-as-shared. This idea remained relatively stable throughout the instructional sequence, most notably when students were solving problems for which either the original net worth or the transaction was unknown. For example, on Day 15 students were asked to solve the problem –17 + ? = +1. Adam said, "18. 17 plus 17 gets you to zero and then I get [go through] zero to one."

Students routinely used the number line to help them structure the gap between two integers, often going through zero to do so. Many students, such as Adam in this example, did not use a physical number line but rather structured the gap mentally by drawing on the imagery of going through zero. The number line and students' taken-as-shared imagery associated with their reasoning on the line afforded them the opportunity to quantify the gap between two integers in meaningful ways and drew them away from the buggy algorithms that they invented at the beginning of instruction.

Practice 4: Reasoning With a VNL to Determine the Results of Addition and Subtraction Operations

Mathematical Practice 4 that emerged within the whole-class discourse was established as students determined the results of a variety of transactions on the original net worth. Here, students routinely interpreted the + and – signs in both unary and binary ways. Three mathematical ideas became taken-as-shared during the establishment of this practice:

- Transactions can have a positive or negative effect on a quantity.
- A vertical number line can be used to find the results of integer operations.
- Subtraction with integers is not commutative.

Transactions can have a positive or negative effect on a quantity. On Day 7 of the instructional sequence, we gave students an activity sheet that listed eight students and decisions they had made about their finances. The intent of this activity was to support students formalizing the effects that operations have on integer net worths into rules for multiplication of integers. For example, Devon added a debt of (–$650) to his net worth statement and Ernie took away a debt of (–$5400) from his. Students were supposed to determine whether each person's decision was good or bad in terms of the effect it had on net worth. It was taken-as-shared almost immediately that when a debt is added or when an asset is taken away, this results in a bad decision. In addition, when an asset is added or a debt is taken away, it is a good decision for the person's net worth. The argumentations on Day 7 indicated that students took these meanings for granted quickly, because there were no warrants or backings requested by anyone. The teacher introduced a way to symbolize these transactions: + (–1000) would mean adding a debt of 1000.

On Day 8, the teacher asked students to determine whether + (+500) and – (–500) were good or bad decisions. Stuart made a conjecture that adding assets and taking away debts are good decisions:

Stuart: A negative outside the parenthesis and inside the parenthesis, you are making money.

Coteacher: You say it is smart?

Stuart: Yes, making money, because you are taking away debt, and it is the same if you have both positive signs. If you have positive outside and negative inside, it is stupid, and if you have negative outside and positive inside, it is stupid, too. [Teacher records this on the board with his words; see Figure 9.]

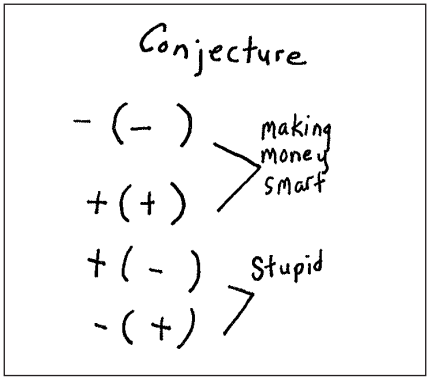

Figure 9. Stuart's conjecture.

Many other students complained that they had the same idea and that the conjecture should have their names on it, too. Argumentations from subsequent class periods indicated that the meaning behind the rules, as shown in Figure 9, were taken-as-shared because students used these rules as data and warrants in subsequent arguments. It was taken-as-shared that a negative applied to a negative and a positive applied to a positive both produce a favorable result. Also, opposite signs generate an unfavorable result.

A VNL can be used to find the results of integer operations. The final two thirds of the instructional sequence built on the taken-as-shared practices with the number line to support students solving integer addition and subtraction problems successfully and with meaning. Students used the number line in flexible ways to prove their calculations were correct. It became taken-as-shared in the discourse that the number line was a viable tool for justifying one's solution, and students used it when their answer was challenged. For instance, on Day 12, the teacher posed the problem 125 + (–225) = ? on the board and almost all students, including Nathan in the dialogue that follows, concluded that the result would be –100:

Nathan: I went 125 down and 100 more and added them up, and it gave me 225 [see Figure 10].

T: Gage, do you want to add anything?

Gage: 125 is the original [net worth] and when you go to zero, there's 100 left.

Dusty: What is the bottom [the circled portion at the lower right in Figure 10]?

Nathan: I went 125 to zero and then zero to 100 and added them up [to get 225].

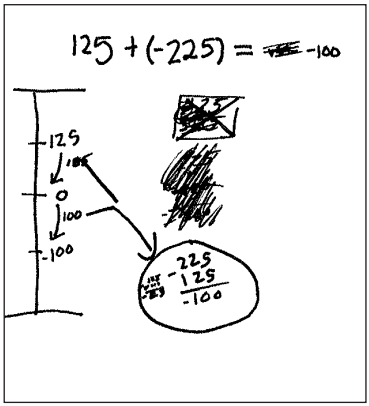

Figure 10. Nathan's number line.

No students challenged the results of this argument, and future discussions in which students were given number sentences of this type indicated that no justifications were needed when students used number lines to express the results of their calculations. Consider the example with Seth, in which students were given the problem –426 + (29). At first, Seth interpreted the problem as "adding a debt" of 29, and symbolized that on the number line with –426 toward the bottom and a downward arrow of 29 (see Figure 11). On reflection, he changed his mind, "There is no negative sign there. Mathematicians are lazy [*sic*] they do not put signs there. So it is –426 to go up 29, so it is –397."

In this example, Seth claimed that the answer was –397. His data were that the net worth is going up 29, and he indicated this with an upward arrow with +29 to its right. This transaction gave him a new net worth of –397. Students neither challenged this nor asked any clarifying questions, indicating that using a VNL to find the result of operations had become taken-as-shared. It was not until we posed number sentences with unknown original net worths and unknown transactions that students' arguments again required warrants and backings.

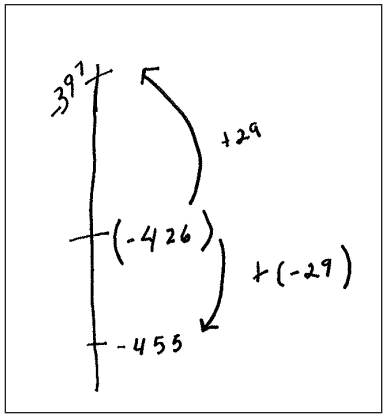

Figure 11. Seth's vertical number line solution.

Subtraction of integers is not commutative. The final mathematical idea that became taken-as-shared as students established the fourth practice involved determining that subtraction of integers is not commutative. This topic came up as a curiosity from one of our students; it was

not intentionally planned as part of the HLT. On Day 16, as students were justifying their answer of –5000 to the problem *unknown* – (–7000) = 2000, Anthony spontaneously asked whether the order of the numbers mattered:

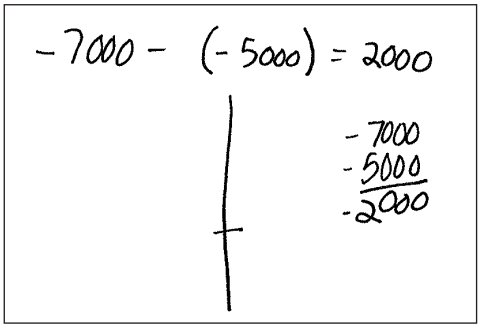

Figure 12. Adam's justification.

T: Let's see. Does it matter? Can you have this –7000 – (–5000) = 2000? Is this going to get you the same result [as –5000 – (–7000)]?

Students: No!

T: Anthony asked a good question. Can we just change the order of these?

Tisha: You get –2000 [for the changed problem].

T: How come?

Adam drew a figure on the board as his proof (see Figure 12).

T: What do you think about that? Questions for him?

Tisha: I want it to be shown on the number line.

T: Tisha, do you want to show it on the number line?

Tisha: Yes. [Draws Figure 13.]

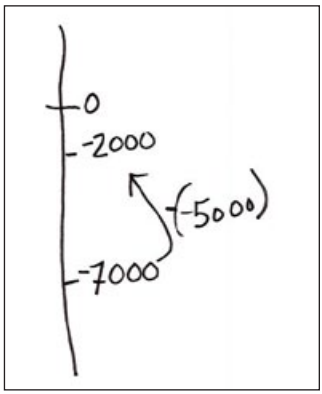

Figure 13. Tisha's verification of –2000.

Bradley: I said that we started with negative 7000 and basically added 5000, but it does not get you up to zero.

The teacher invited students to continue to explore this relationship by giving them two problems, the numbers of which had been switched. Students concluded that they were not the same, but Tisha added:

> It works for addition because it does not matter how you switch the numbers. Like you can do 7 + 5 and 5 + 7 and you get the same answer. But when you are doing subtracting and you have a lower number or a bigger number you cannot switch them because if you did 7 – 5 you get 2, but if you do 5 – 7 you totally get a different answer.

The teacher then referred to commutative properties and told students that they had just discovered that subtraction is not commutative.

Practice 5: Determining the Meaning of Positive and Negative Signs

At the outset of the experiment, we had designed instruction so that students

progressed from solving problems in the context of assets, debts, and net worths to modeling integer operations with a VNL. The instructional sequence intentionally began by building on students' situation-specific imagery (that imagery associated with a real-world context) and moved them little by little to reasoning with abstract symbols. We now describe Mathematical Practice 5 that emerged as students determined the meaning of these abstract symbols. Two mathematical ideas became taken-as-shared in the constitution of this practice:

- Different operations (transactions) can have the same effect on a quantity.
- A minus sign is different from a negative sign.

Different operations (*transactions*) *can have the same effect on a quantity.* Reasoning with a number line to find the result of a transaction had become taken-as-shared in classroom Mathematical Practice 4. Now, students were presented with number sentences in which the transaction was unknown. On Day 13, the teacher posed the number problem +10,000 _____ = 7000, and students were asked to fill in the blank with an appropriate transaction. Students' solutions to this problem were + (–3000), – (+3000), – (–3000) [the teacher's contribution], –(+800) – (+2200), – (+1000) + (–2000), and + (–3100) + (100). Students immediately rejected – (–3000) saying, "Because then he gains" and "We're supposed to go down." The only two solutions that needed additional support in the form of data and warrants were – (+1000) + (–2000) and + (–3100) + (100). It became taken-as-shared here that there are multiple correct transactions that could result in a $7000 net worth, and students began to think flexibly and creatively on these tasks.

Consider another example in which students were given a net worth of $10,000 and asked what transactions could result in a new net worth of $12,000. The following is a list of all the transactions students created: + (+2000), – (–2000), – (+3000) + (+5000), – (+1000) + (+3000), and + (–5000) + (+7000). No warrants or backing were required to justify these transactions, and except for – (–2000) and – (+3000) + (+5000), students did not call for data to justify their conclusions.

A minus sign is different from a negative sign. The final mathematical idea that became taken-as-shared in the constitution of Mathematical Practice 5 involved making sense of the "–" symbol. As pointed out in the literature review, students interpret the "–" and "+" signs as both an operator (action) and a state. This was true of our students as well, especially since we chose to use two symbols in writing our transactions, for example, + (+100). In this case, it was taken-as-shared that the first sign stood for the action of adding and the second symbolized a state (an asset). We confronted this duality with students by posing problems with only one symbol: 7 – 9 and –7 – 9:

Anthony: Minus 2 [for 7 – 9].
Students: Negative 2!

T: Is it different, minus 2 and negative 2? What is the difference Charlie?

Charlie: The difference is that when you minus something, you are taking away, and negative is your debt. It is totally different.

T: All right. We need to sort it out. Some people think that minus 2 and negative 2 are different.

Student: Minus is an operation and negative is an amount.

T: Minus 2 is an operation like you take away, to do some transaction.

Gage: Wait, never mind! They are the same. Because when you draw something like – (–2), you take away a debt.

The responses given in this excerpt underscore the difficulty students had with interpreting the functions of the integer sign when only one was present. There was a clear difference for most students until Gage concluded that minus and negative were the same. The conversation took a different turn in the class at this point, but we raised the issue again on Day 18 when students were asked to determine whether the expression – (–155 – 5) was equivalent to – (–150). Stuart said, "You have a minus 155 and you just add a plus sign here [in front of the 5] since there is no sign. The sign between the 155 and the 5 does not mean negative, it is a minus." Mark suggested that it is like taking away an asset of 5. Stuart even rewrote part of the problem as –155 – (+5). Students indicated agreement with Stuart and Mark and began to deal with single-signed problems by rewriting them with two signs. Our discourse analysis led us to conclude that it was taken-as-shared that the "–" symbol can stand for the operation of take away as well as a state (a debt). If students had difficulty interpreting the function of the sign, then they typically inserted a positive sign after the "–" sign to indicate that it was a positive number that was being subtracted.

DISCUSSION

Our goal in this experiment was to test and revise an HLT for integer addition and subtraction to propose a potentially viable instructional theory. We drew on the work of other researchers who investigated students' understanding of negative numbers and explored various models and contexts for teaching integers. Upon review of the literature and drawing on our research experiences in other mathematical domains, we decided to merge the context of finance with a vertical empty number line to explore the possibility that this blend could overcome some of the limitations identified in previous research.

Our findings suggest that grounding students' integer work in the context of finance, coupled with a VNL, has great potential. Our evidence comes from our analysis of the mathematical learning of the classroom community in the form of five mathematical practices as well as the results of pretests and posttests. The classroom mathematical practices indicate that students can construct conceptual understandings of integers and their operations. First, students in our study were able to reason about a number "going below zero" almost immediately. Second, the

concept that an integer is an abstract object emerged for students as they established Mathematical Practice 1. It became taken-as-shared that a net worth was an integer value that signified the difference in one's total assets and debts. The construct of net worth supported students' interpretations of –5 – (–10) later in the sequence. For instance, students would readily interpret –5 as a person's net worth and could imagine taking away a debt of 10 from it. Such a problem was difficult for students to interpret in previous years because students thought that –5 signified a debt of 5 and could not imagine taking away a debt of 10 from a debt of 5. Postinterviews also show growth, in that no one changed the problem to 10 – 5 to get 5, as in the preinterviews.

Mathematical Practice 2 suggested how powerful it was, in this context, for students to imagine referencing zero to handle the relationship between two integers. Paying off became a strong, student-generated metaphor for determining and comparing two net worths (integers), for finding the results of operations, and for cancelling two values (additive inverses). Mathematical Practice 2 is consistent with Peled et al.'s (1989) findings that students naturally want to organize their work using zero as a benchmark, particularly on a number line. Our students created zero as a benchmark in the form of payoff long before being introduced to the number line, but inscribed their thinking on the VNL quite naturally.

The HLT was written to support both unary and binary interpretations and as students established Mathematical Practices 4 and 5, both interpretations were prevalent. Students reasoned with signs in both a unary and binary function when they had to solve problems such as –1000 – (+900) = *unknown, unknown* + (–100) = 50, and 600 *unknown transaction* = 500. In the first two problems, the two signs in the middle of the problem functioned as both a state and an operator (Streefland, 1996). Mathematical Practice 5 indicated that students interpreted the two signs in problems of the form $a + (-b)$ as, "+" means add and "–" means negative. Although this is what we were attempting to support, students had difficulty with problems posed in the form $a - b$, with $b > a$. Because a binary interpretation was so strongly supported, students would ask questions such as, "Is that a subtraction sign or a negative?" Future instructional activities will need to confront students' interpretation of the quantities when only one sign is in the middle (e.g., –50 – 50).

Finally, although almost all the studies on integers report that students had difficulty subtracting negative numbers, *t* tests run on our students' pretests and posttests indicated that the instructional sequence supported students' development of integer addition and subtraction, both procedurally and conceptually. Students improved their understanding of both addition and subtraction through this sequence, but the improvement was especially significant in their understanding of subtraction. The tests contained four questions on addition and subtraction with two questions being conceptual and two procedural. The two conceptual questions were awarded 0 points for completely incorrect explanations and 4 points for completely correct ones, with partially correct explanations assigned 1 to 3 points. The two procedural questions, on the other hand, were awarded 2 points for a completely correct answer, 1 point for a correct answer but incorrect sign (or vice versa), and 0 points for both

incorrect figure and incorrect sign. The first procedural question was composed of a single part (thus contributing a maximum of 2 points), and the second procedural question was composed of 10 parts (contributing to a maximum of 20 points). With this grading scheme, scores on each test could range from 0 to 30. The reliability of the tests was established in a previous study (Smith, 1995).

A paired-samples *t* test was conducted to compare students' achievement in these pretests and posttests. We found a significant difference between the test scores of the students before and after the instruction, as summarized in Table 3. These results indicate that—

1. the instruction helped students significantly improve their test scores for both addition and subtraction.
2. although the improvement in subtraction significantly surpassed the improvement in addition, the students' achievement in addition remained significantly better than their achievement in subtraction; $t(17) = 3.01$, $p = .008$.

There are several revisions we would make to the sequence based on our analysis of the collective learning of this classroom. As previously stated, we would build in activities to specifically confront the dual nature of the negative sign to help students better interpret problems that contain only one middle sign. Second, the VNL first emerged as a model of students determining a person's net worth. We had constructed the HLT so that the VNL would emerge later in the sequence, as a model of students' transactions. Since the VNL emerged much earlier, we would like to explore ways in which it could signify students' strategies for a multitude of problems, including determining the net worth (putting total assets, for example, on the line and bringing it down by the total debts), comparing net worths (difference between two net worths), as well as operations (transactions on one net worth).

A third revision suggested by our analysis involves the question of transfer. Do our students transfer their understanding of integers to other contexts? Interviews and posttests only insufficiently answered this question possibly affirmatively. Students were given context-free number problems and performed well on them (indicating that vertical mathematization was a success). However, horizontal mathematization, or work with integers across other contexts, is still in question.

Table 3
Paired t-test Results for Students' Scores in Pretests and Posttests

	Pretest (addition)	Posttest (addition)	Pretest (subtraction)	Posttest (subtraction)
Mean	20.89	26.11	6.28	23.72
Std. Dev.	6.62	3.45	3.12	4.06
$t(17)$	–5.15		–15.25	
p	< 0.001		< 0.001	

Intuitively, results on students' classroom unit test used for grading purposes indicate that students could reason successfully in temperature, altitude, and other contexts. We also would like to explore extensions of this context to multiplication and division of integers.

REFERENCES

Akyuz, D. (2010). *Supporting a standards-based teaching and learning environment: A case study of an expert middle school mathematics teacher*. (Doctoral Dissertation). University of Central Florida, Orlando, FL.

Battista, M. T. (1983). A complete model for operations on integers. *Arithmetic Teacher, 30*(9), 26–31.

Bauersfeld, H. (1992). Classroom cultures from a social constructivist perspective. *Educational Studies in Mathematics, 23*, 467–481. doi:10.1007/BF00571468

Blumer, H. (1969). *Symbolic interactionism: Perspective and method.* Englewood Cliffs, NJ: Prentice-Hall.

Bofferding, L. (2010). Addition and subtraction with negatives: Acknowledging the multiple meanings of the minus sign. In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 703–710). Columbus, OH: The Ohio State University.

Cobb, P. (2003). Investigating students' reasoning about linear measurement as a paradigm case of design research. In N. Pateman (Series Ed.), M. Stephan, J. Bowers, & P. Cobb (with K. Gravemeijer) (Vol. Eds.), *Journal for Research in Mathematics Education Monograph Series: Vol. 12. Supporting students' development of measuring conceptions: Analyzing students' learning in social context* (pp. 1–16). Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9–13. doi:10.3102/0013189X032001009

Cobb, P., Gravemeijer, K., Yackel, E., McClain, K., & Whitenack, J. (1997). Mathematizing and symbolizing: The emergence of chains of signification in one first-grade classroom. In D. Kirshner & J. A. Whitson (Eds.), *Situated cognition: Social, semiotic, and psychological perspectives* (pp. 151–233). Mahwah, NJ: Erlbaum.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*, 175–190. doi:10.1080/00461520.1996.9653265

Davis, R. B. (1967). *Explorations in mathematics: A text for teachers*. Palo Alto, CA: Addison-Wesley.

De Bock, D., Deprez, J., Van Dooren, W., Roelens, M., & Verschaffel, L. (2011). Abstract or concrete examples in learning mathematics? A replication and elaboration of Kaminski, Sloutsky, and Heckler's study. *Journal for Research in Mathematics Education, 42*, 109–126. http://www.nctm.org/publications/jrme.aspx

De Lange, J. (1987). *Mathematics, insight and meaning: Teaching, learning and testing of mathematics for the life and social sciences*. Utrecht, the Netherlands: Vakgroep Onderzoek Wiskundeonderwijs en Onderwijs Computercentrum, Rijksuniversiteit Utrecht.

Dirks, M. K. (1984). The integer abacus. *Arithmetic Teacher, 31*(7), 50–54.

Forman, E., Larreamendy-Joerns, J., Stein, M., & Brown, C. (1998). "You're going to want to find out which and prove it": Collective argumentation in a mathematics classroom. *Learning and Instruction*, *8,* 527–548. doi:10.1016/S0959-4752(98)00033-4

Fosnot, C. T., & Dolk, M. (2005). "Mathematics" or "mathematizing"? In C. T. Fosnot (Ed.), *Constructivism: Theory, perspectives and practice* (2nd ed., pp. 175–191). New York, NY: Teachers College Press.

Freudenthal, H. (1968). Why to teach mathematics so as to be useful. *Educational Studies in Mathematics 1*, 3–8. doi:10.1007/BF00426224

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht, the Netherlands: D. Reidel.

Gallardo, A. (2002). The extension of the natural-number domain to the integers in the transition from arithmetic to algebra. *Educational Studies in Mathematics, 49*, 171–192. doi:10.1023/A:1016210906658

Gallardo, A., & Rojano, T. (1994). School algebra: Syntactic difficulties in the operativity with negative numbers. In D. Kirshner (Ed.), *Proceedings of the Sixteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 159–165). Baton Rouge, LA: Louisiana State University.

Glaeser, G. (1981). Epistémologie des nombres relatifs. *Recherches en Didactique des Mathématiques, 2*, 303–346. http://rdm.penseesauvage.com/

Gravemeijer, K. (1994). *Developing realistic mathematics education*. Utrecht, the Netherlands: CD-Press.

Gravemeijer, K., Bowers, J., & Stephan, M. (2003). Continuing the design research cycle. In N. Pateman (Series Ed.), M. Stephan, J. Bowers, & P. Cobb (with K. Gravemeijer) (Vol. Eds.), *Journal for Research in Mathematics Education Monograph Series: Vol. 12. Supporting students' development of measuring conceptions: Analyzing students' learning in social context* (pp. 103–169). Reston, VA: National Council of Teachers of Mathematics.

Gravemeijer, K., & Stephan, M. (2002). Emergent models as an instructional design heuristic. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 145–169). Dordrecht, the Netherlands: Kluwer.

Hefendehl-Hebeker, L. (1991). Negative numbers: Obstacles in their evolution from intuitive to intellectual constructs. *For the Learning of Mathematics, 11*(1), 26–32.

Janvier, C. (1985). Comparison of models aimed at teaching signed integers. In L. Streefland (Ed.), *Proceedings of the Ninth International Conference for the Psychology of Mathematics Education* (pp. 135–140). Utrecht, the Netherlands: State University of Utrecht.

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2008). The advantage of abstract examples in learning math. *Science, 320*, 454–455. doi:10.1126/science.1154659

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb and H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229–270). Hillsdale, NJ: Erlbaum.

Lappan, G., Fey, J., Fitzgerald, W., Friel, S., & Phillips, E. (1998). *Connected mathematics project*. Palo Alto, CA: Dale Seymour.

Liebeck, P. (1990). Scores and forfeits: An intuitive model for integer arithmetic. *Educational Studies in Mathematics, 21,* 221–239.

Linchevski, L., & Williams, J. (1999). Using intuition from everyday life in "filling" the gap in children's extension of their number concept to include the negative numbers. *Educational Studies in Mathematics, 39*, 131–147. doi:10.1023/A:1003726317920

Lytle, P. A. (1994). Investigation of a model based on the neutralization of opposites to teach integer addition and subtraction. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the 18th International Conference for the Psychology of Mathematics Education* (Vol. III, pp. 192–199). Lisbon, Portugal: University of Lisbon.

Mukhopadhyay, S., Resnick, L. B., & Schauble, L. (1990). Social sense-making in mathematics: Children's ideas of negative numbers. In G. Booker, J. Cobb, & T. N. de Mendicuti (Eds.), *Proceedings of the 14th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 281–288). Oaxtapec, Mexico: PME.

Murray, J. C. (1985). Children's informal conceptions of integer arithmetic. In L. Streefland (Ed.), *Proceedings of the Ninth International Conference for the Psychology of Mathematics Education* (pp. 147–153). Utrecht, the Netherlands: The State University of Utrecht.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Peled, I., Mukhopadhyay, S., & Resnick, L. B. (1989). Formal and informal sources of mental models for negative numbers. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education* (Vol. III, pp. 106–110). Paris, France: PME.

Rasmussen, C., & Stephan, M. (2008). A methodology for documenting collective activity. In A. E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering and mathematics learning and teaching* (pp. 195–215). New York, NY: Routledge.

Rasmussen, C., Stephan, M., & Allen, K. (2004). Classroom mathematical practices and gesturing. *Journal of Mathematical Behavior, 23*, 301–323. doi:10.1016/j.jmathb.2004.06.003

Schwarz, B. B., Kohn, A. S., & Resnick, L. B. (1993/1994). Positives about negatives: A case study of an intermediate model for signed numbers. *The Journal of the Learning Sciences, 3*, 37–92. doi:10.1207/s15327809jls0301_2

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114–145. doi:10.2307/749205

Smith, J. P. (1995). *The effects of a computer microworld on middle school students' use and understanding of integers*. (Unpublished doctoral dissertation). The Ohio State University, Columbus, Ohio. Retrieved from http://rave.ohiolink.edu/etdc/view?acc_num=osu1248798217

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Hillsdale, NJ: Erlbaum.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. New York, NY: Praeger.

Stephan, M. L. (2003). Reconceptualizing linear measurement studies: The development of three monograph themes. In N. Pateman (Series Ed.), M. Stephan, J. Bowers, & P. Cobb (with K. Gravemeijer) (Vol. Eds.), *Journal for Research in Mathematics Education Monograph Series: Vol. 12. Supporting students' development of measuring conceptions: Analyzing students' learning in social context* (pp. 17–35). Reston, VA: National Council of Teachers of Mathematics.

Stephan. M. L. (2009). What are you worth? *Mathematics Teaching in the Middle School, 15*, 16–23. http://www.jstor.org/stable/41182947

Stephan, M., Bowers, J., Cobb, P., & Gravemeijer K. (Eds.). (2003). *Journal for Research in Mathematics Education Monograph Series: Vol. 12. Supporting students' development of measuring conceptions: Analyzing students' learning in social context* (N. Pateman, Series Ed.). Reston, VA: National Council of Teachers of Mathematics.

Stephan, M., Cobb, P., Gravemeijer, K., & Estes, B. (2001). The role of tools in supporting students' development of measuring conceptions. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics*, 2001 Yearbook of the National Council of Teachers of Mathematics (NCTM) (pp. 63–76). Reston, VA: NCTM.

Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *Journal of Mathematical Behavior, 21*, 459–490. doi:10.1016/S0732-3123(02)00145-1

Streefland, L. (1996). Negative numbers: Reflections of a learning researcher. *Journal of Mathematical Behavior, 15*, 57–79. doi:10.1016/S0732-3123(96)90040-1

Thompson, P. W., & Dreyfus, T. (1988). Integers as transformations. *Journal for Research in Mathematics Education, 19*, 115–133. doi:10.2307/749406

Toulmin, S. E. (1958). *The uses of argument*. Cambridge, England: Cambridge University Press.

Vlassis, J. (2004). Making sense of the minus sign or becoming flexible in 'negativity'. *Learning and Instruction, 14*, 469–484. doi:10.1016/j.learninstruc.2004.06.012

Vlassis, J. (2008). The role of mathematical symbols in the development of number conceptualization: The case of the minus sign. *Philosophical Psychology, 21*, 555–570. doi:10.1080/09515080802285552

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Bristol, PA: The Falmer Press.

Authors

Michelle Stephan, University of North Carolina, Center for STEM Education, 9201 University City Boulevard, Charlotte, NC 28223-0001; michelle.stephan@uncc.edu

Didem Akyuz, Middle East Technical University, Faculty of Education, Department of Elementary Education, Ankara, Turkey; dakyuz@metu.edu.tr

Accepted December 19, 2011