CASE 4

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Slippery Cylinders

PRE-CASE EXERCISES

Please complete the following activity: Create two paper cylinders from identical pieces of paper by rolling them lengthwise and widthwise. Answer the following questions:

1. Which one do you think would hold more water? How would you convince someone else?
2. What understanding does your demonstration require you, as a learner, to have?
3. What connections does this exercise have to other mathematical ideas (e.g., limits, rates of change, min./max., etc.)?

THE CASE

"I am amazed!" Heather Lister said to her friend and colleague, Carolyn Jenkins, who had just observed her class. "It was just as you predicted," she continued as she shook her head.

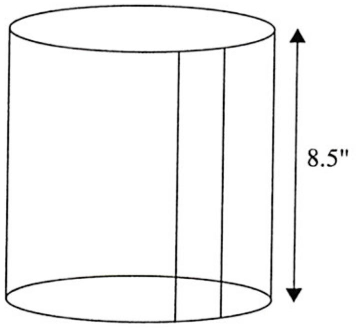
"Actually, almost all of them made mistakes, but not the same mistakes, and not for the same reasons," commented Carolyn.

The Experiment

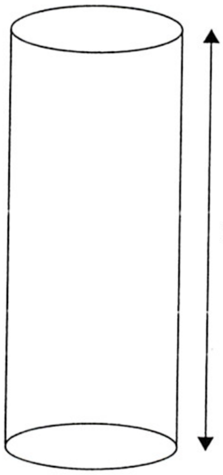
A week earlier, Ms. Lister had had dinner with Mrs. Jenkins, who was a supervisor for student teachers at a neighboring school. Mrs. Jenkins had described to her colleague an experiment in geometry she did with a group of her student teachers. "You take a piece of regular paper and you roll it this way, so that the 8½ inch sides meet like this," Mrs. Jenkins had demonstrated while rolling up a sheet of notebook paper so that the sides just met, with no overlap. "Then you take the same piece of paper and roll it the other way so that the 11 inch sides meet (see Figure 4.1). How will the volumes of the cylinders compare?"

"What did your student teachers think?" Ms. Lister asked. "My juniors are studying a unit on solids and volumes right now. They're pretty bright. I think they would know the answer just by looking at the problem, without doing anything."

Figure 4.1 Mrs. Jenkins' Demonstration



Circumference = 11"



Circumference = 8.5"

"Try it out with your class," Mrs. Jenkins urged. "I'd love to see how they deal with this problem. In fact, I'd like to come watch what they do with it. Would you mind?"

"Not at all," said Ms. Lister replied. "In another week we'll be studying cylinders and cones, so this will fit perfectly into my curriculum! I wasn't planning to do any formal comparison of surface area and volume so they won't have had any experience doing that, but I don't think it really matters. It's a good concept for them to explore." The two agreed upon a time when Mrs. Jenkins would come to observe Ms. Lister's students tackle the problem.

The Class

Ms. Lister loved her junior math class. She found them to be eager, cooperative learners. It helped that she had worked with more than half of them in previous years, since she taught ninth- and tenth-grade math as well. They seemed to trust her and were willing to try whatever she asked them to do. Ms. Lister was known as a good teacher, by colleagues and students alike. Student teachers and other observers were regular additions to the classroom as well, so it wasn't any surprise when Mrs. Jenkins showed up.

On a Tuesday morning in February, Mrs. Jenkins came into Ms. Lister's room just before class began and sat at a desk at the side of the room. After the bell rang, Ms. Lister called to her class, "Okay, folks! Let's get started. Remember I told you yesterday that we would have a visitor? I'd like to introduce my friend, Mrs. Jenkins. She has come to watch today so that she can help me with a project I am doing. If you have any questions later when we are working on our classwork assignment, you can ask either one of us." The students glanced briefly in her direction, then turned their attention to the homework solution sheets that their teacher was passing out. "First we will go over problems from the homework. Then we will break up into groups and try an experiment having to do with volume."

"I'd like you to get into your groups now," Ms. Lister announced after the homework review. She often varied how she assigned students to groups during the year. These particular groups were formed on the basis of common interest. Each group was working on a term project on some aspect of mathematics. She assigned partners based on a survey that she gave out at the beginning of the term asking what area of math each student was interested in studying as an outside project (biography of a mathematician, math in art, non-Euclidean geometry, computer applications, history of math, and so forth). So, the groups were neither gender- nor ability-balanced.

The Problem

Students got up and started moving desks around the room, positioning themselves into groups of twos and threes. Once they were all seated, Ms. Lister gave each group two pieces of 8½" × 11" paper, with the dimensions clearly written on the edges of the pages. "May I have your attention," she called out. "Each group should now have two pieces of 8½" × 11" paper." She also pointed to a pile of supplies on her desk: tape, scissors, string, and rulers. "You may use whatever supplies you need as you work on this problem," she told the class. "I'm going to hand out the instructions. Please follow them carefully." Ms. Lister then gave the students the instruction sheet shown in Figure 4.2.

Mrs. Jenkins was anxious to hear what students thought about the problem before they did any calculations. Trying to be as neutral in demeanor and voice as possible, she asked the students in group after group, "What do you think? Will the volumes be the same or different?"

Jim, a young man who was eager to please adults, said, "I think they're the same, but Ms. Lister wouldn't have us go through this if it were that simple." He turned to his groupmates and said, "Let's try to figure it out."

"I can't tell until I do the calculations," Jonathan, a quiet young man, replied.

The rest of the students in group after group gave Mrs. Jenkins the same answer. "Of course they will be the same!" Several students added, "We're using the same size piece of paper!"

Ms. Lister also circulated around the room, glancing at each paper. Amber's group was stuck trying to figure out how to calculate the surface area. "Unroll one of your cylinders and look at what you have," she suggested.

Amber picked at the tape she had used to attach the sides of her paper to form a cylinder and unrolled it. "Oh! I see, it's just a rectangle. I'm so

Figure 4.2 Ms. Lister's Assignment

Cylinder Exploration (613)

Use your 8.5" x 11" paper to help your group visualize the following exercises.

1. Roll one paper the long way to make a cylinder. Tape it together with no overlap.
2. Roll a second paper the short way to make a cylinder. Tape it together with no overlap.
3. Calculate the lateral surface areas of both cylinders (LSA).

LSA long cylinder = ____________________

LSA short cylinder = ____________________

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4. As a *group* decide which of the following statements is true. Go by your intuition. You may discuss your ideas, but please do not do any computations yet. Pick one, and write a short paragraph explaining why.

   (a) Long cylinder has greater volume.

   (b) Short cylinder has greater volume.

   (c) The volumes are the same.

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5. Calculate the volumes of each of the cylinders. You may use any tools you like.

Volume long cylinder = ____________________

Volume short cylinder = ____________________

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6. Please write a summary statement which relates the lateral surface area and the volume of a cylinder. Do you think you could make any generalizations concerning the relationship between lateral surface area and volume of other containers?

stupid sometimes," Amber responded with a giggle as she turned back to her group. "I get it now."

As students calculated the volumes, they began to murmur to each other: "I don't understand!," "We must have done something wrong," "They should be the same."

Paper Airplanes

In one corner of the room Kevin and Chuck were making paper airplanes using the sheets of paper. Mrs. Jenkins asked them if they were done. "Oh yeah," said Kevin, "we proved they had the same volumes!" Chuck added proudly, "We didn't even have to use the tape!"

"How did you prove it?" asked Mrs. Jenkins.

"Well, here it is," said Chuck, showing his calculations on a piece of paper (see Figure 4.3). "The volume is perimeter times height, so in the case of the taller one it is 8½" × 11". In the case of the short, fat one it is 11" × 8½". It all comes out the same. We were right to begin with." Something in Mrs. Jenkins' expression made Kevin uneasy. "Right?" he asked hopefully, looking up with wide eyes.

"Wrong," she said.

"Ohhhh kaaaay," said Chuck dramatically, "let's look at it again."

"I told you that was too easy," Kevin mumbled in Chuck's direction.

Steven, Dina, and Bob

Ms. Lister walked by Steven, Dina, and Bob. Steven was doodling on his paper, and Dina and Bob were chatting about what they had done the previous weekend. Ms. Lister worried about this group. Although generally cooperative, Bob had a real attitude sometimes. He only did his homework from time to time and was very social in this class. As she glanced at their papers, she noticed that they had written the same number down for the volumes of the cylinders (see Figure 4.4). "Did you really get the same thing for both of these?" she asked the three students.

Steven responded, "Yep!"

"Where are your calculations?" Ms. Lister pursued.

Bob pointed to his paper. "Here."

"But I only see one set of computations on your scrap paper. Where's the other?" Ms. Lister inquired.

"We didn't do it *twice*. That would have been a waste of time since they're going to be the same!" Bob explained.

"Don't be so sure," she said. "You really need to do the *entire* worksheet before you make any conclusions." Ms. Lister tapped on Bob's paper and waited for the three students to move back together and begin to work before she moved away from the group. As she left she heard Bob mumble, "This is so stupid. They're going to be the same. Any idiot can see that! What a waste of time!"

Discouragement

From across the room Ms. Lister could see Amber, Maggie, and Hope sitting in their seats silently staring at their papers (see Figure 4.5). As she walked closer she could see that they were quite frustrated. "What's wrong, Amber?"

"We keep doing this over and over again and we keep getting different numbers for the volumes. We can't figure out what we're doing wrong!" Amber said with an edge in her voice.

Ms. Lister looked at the other members of the group. They were nodding in agreement.

"What makes you think you're doing anything wrong?" she asked.

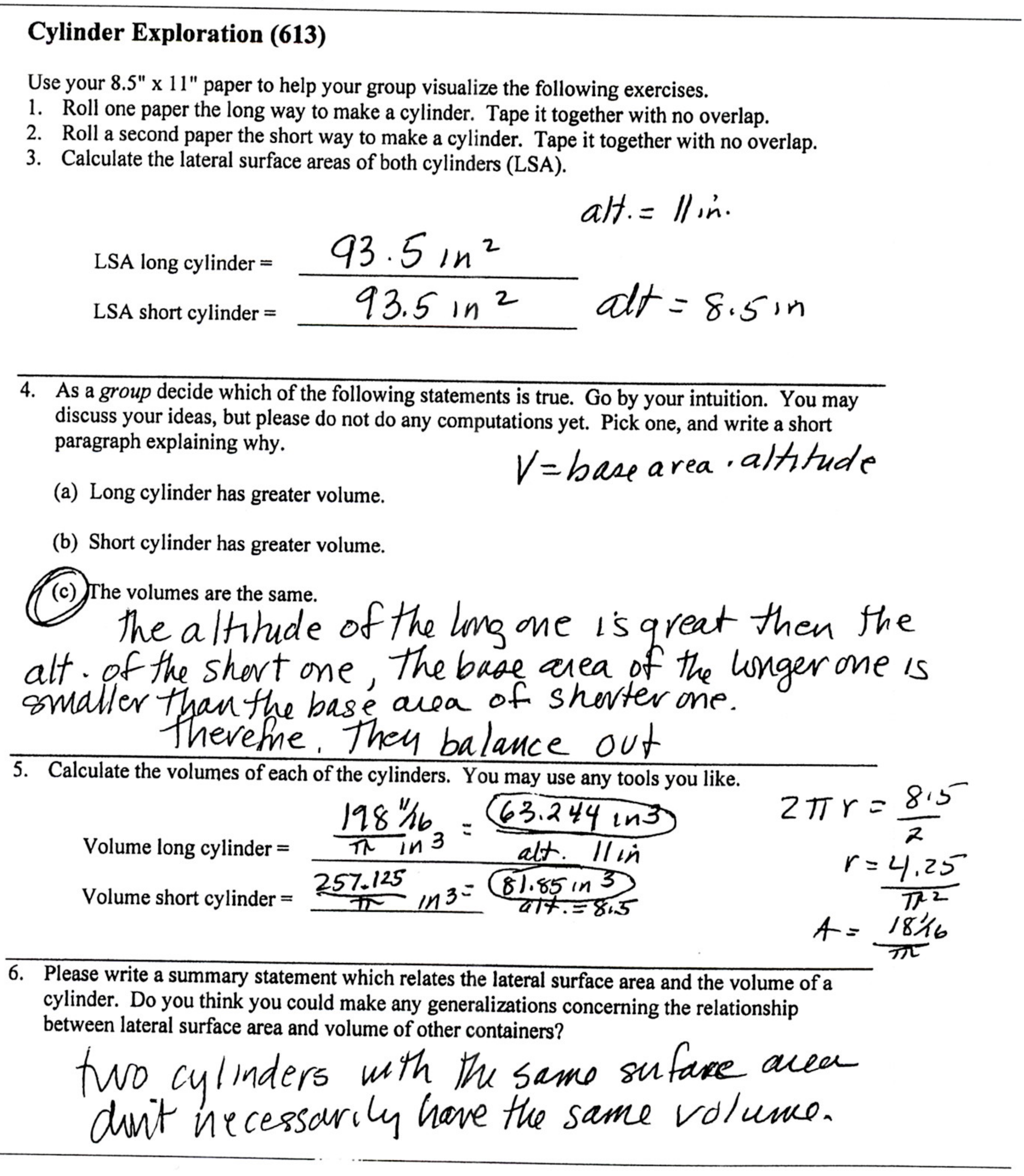
Ms. Lister could see that the girls were about to give up. "You're on the right track. You did all your calculations perfectly! The volumes are *not* the same. Your job now is to figure out why."

Understandings

After about 10 minutes, most groups had discovered that there was indeed a difference in the two volumes. Ms. Lister was pleased to hear some students reasoning out why it made sense that the volumes might be different and discussing it with each other. Although a few students were tenaciously arguing that the volumes *had* to be the same, their classmates were doing a good job of explaining why they weren't.

Kelly, Pete, and Sergio had completed their worksheet and were doing some interesting experiments. They cut one piece of paper in two halves lengthwise so that each had dimensions of 4¼" × 11", taping together the two pieces to create a long, thin tube (see Figure 4.6). "What will the volume of this cylinder be?" they asked Ms. Lister, who was watching them work.

Figure 4.3 Kevin and Chuck's Work



Cylinder Exploration (613)

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1. Roll one paper the long way to make a cylinder. Tape it together with no overlap.
2. Roll a second paper the short way to make a cylinder. Tape it together with no overlap.
3. Calculate the lateral surface areas of both cylinders (LSA).

alt. = 11 in.

LSA long cylinder = 93.5 in^2

LSA short cylinder = 93.5 in^2 alt = 8.5 in

4. As a *group* decide which of the following statements is true. Go by your intuition. You may discuss your ideas, but please do not do any computations yet. Pick one, and write a short paragraph explaining why.

V = base area · altitude

(a) Long cylinder has greater volume.

(b) Short cylinder has greater volume.

[✓ (c)] The volumes are the same.

The altitude of the long one is great then the alt. of the short one, the base area of the longer one is smaller than the base area of shorter one. Therefore. They balance out

5. Calculate the volumes of each of the cylinders. You may use any tools you like.

Volume long cylinder = $\frac{198\,^{11}/_{16}}{\pi} \text{ in}^3 = \boxed{63.244 \text{ in}^3}$ alt. 11 in

Volume short cylinder = $\frac{257.125}{\pi} \text{ in}^3 = \boxed{81.85 \text{ in}^3}$ alt. = 8.5

$2\pi r = \frac{8.5}{2}$

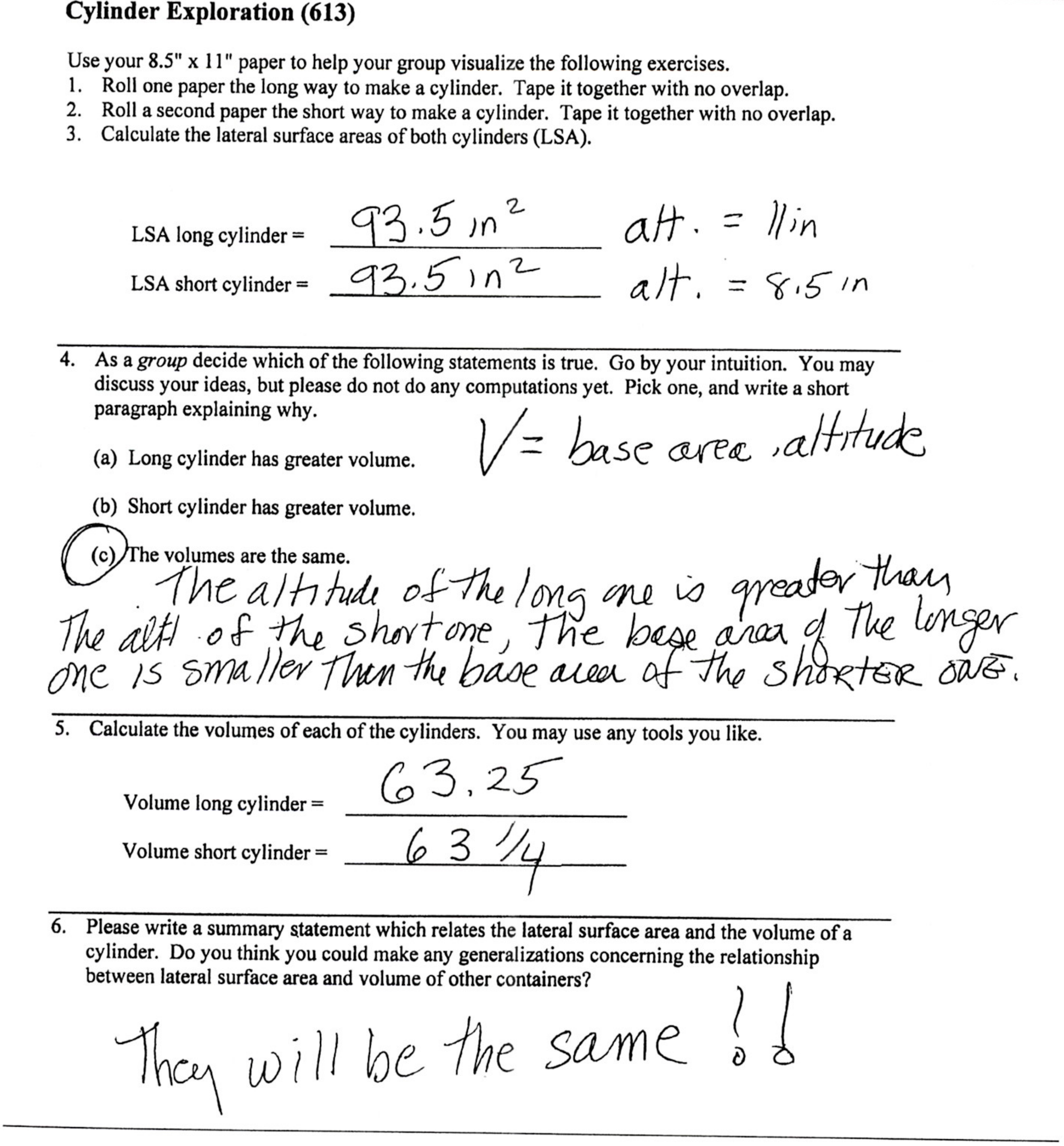
$r = \frac{4.25}{\pi^2}$

$A = \frac{18\,^{1}/_{16}}{\pi}$

6. Please write a summary statement which relates the lateral surface area and the volume of a cylinder. Do you think you could make any generalizations concerning the relationship between lateral surface area and volume of other containers?

two cylinders with the same surface area don't necessarily have the same volume.

Figure 4.4 Steve, Dina, and Bob's Work



Cylinder Exploration (613)

Use your 8.5" x 11" paper to help your group visualize the following exercises.

1. Roll one paper the long way to make a cylinder. Tape it together with no overlap.
2. Roll a second paper the short way to make a cylinder. Tape it together with no overlap.
3. Calculate the lateral surface areas of both cylinders (LSA).

LSA long cylinder = 93.5 in^2 alt. = 11in

LSA short cylinder = 93.5 in^2 alt. = 8.5 in

4. As a *group* decide which of the following statements is true. Go by your intuition. You may discuss your ideas, but please do not do any computations yet. Pick one, and write a short paragraph explaining why.

V = base area · altitude

(a) Long cylinder has greater volume.

(b) Short cylinder has greater volume.

[✓ (c)] The volumes are the same.

The altitude of the long one is greater than The alt. of the short one, The base area of the longer one is smaller Then the base area of the shorter one.

5. Calculate the volumes of each of the cylinders. You may use any tools you like.

Volume long cylinder = 63.25

Volume short cylinder = $63\frac{1}{4}$

6. Please write a summary statement which relates the lateral surface area and the volume of a cylinder. Do you think you could make any generalizations concerning the relationship between lateral surface area and volume of other containers?

They will be the same !!

Figure 4.5 Amber, Hope, and Maggie's Work

Cylinder Exploration (613)

Use your 8.5" x 11" paper to help your group visualize the following exercises.

1. Roll one paper the long way to make a cylinder. Tape it together with no overlap.
2. Roll a second paper the short way to make a cylinder. Tape it together with no overlap.
3. Calculate the lateral surface areas of both cylinders (LSA).

LSA long cylinder = $8.5 \cdot 11 = 93.5 \text{ in}^2$

LSA short cylinder = $11 \cdot 8.5 = 93.5 \text{ in}^2$

---

4. As a *group* decide which of the following statements is true. Go by your intuition. You may discuss your ideas, but please do not do any computations yet. Pick one, and write a short paragraph explaining why.

(a) Long cylinder has greater volume.

(b) Short cylinder has greater volume.

[✓ (c)] The volumes are the same.

We think The volumes will be the same. One has a bigger altitude but smaller base area. The other has a smaller altitude but bigger base area. We think That multiplying the base area times the altitude for each cyclinder will result in the same product

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5. Calculate the volumes of each of the cylinders. You may use any tools you like.

Volume long cylinder = $11 \cdot (\pi 1.4^2) = 21.56\pi$ in³

Volume short cylinder = $8.5 \cdot (\pi 1.8^2) = 27.54\pi$

$C = 8.5 \quad 2\pi r = 8.5, \ r \approx 1$

$C = 11, \ 2\pi r = 1, \ r \approx 1/4$

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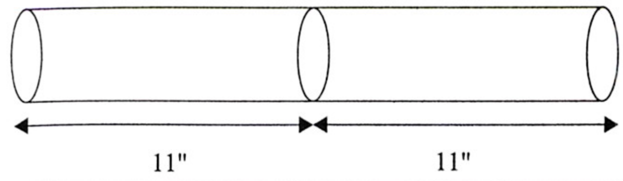
6. Please write a summary statement which relates the lateral surface area and the volume of a cylinder. Do you think you could make any generalizations concerning the relationship between lateral surface area and volume of other containers?

two cylinders That have the same lateral surface area do not necessarily have the same volume.

Figure 4.6 Kelly, Pete, and Sergio's Work



"You figure it out," she replied.

The group next to them followed their lead, but made a short, fat cylinder. They, too, wanted to be told if theirs was going to have a bigger or smaller volume than either of the originals, but Ms. Lister refused (they knew she would).

As time began to run out, Ms. Lister asked, "Could I have everyone's attention up front, please? I'd like to see a show of hands. Who was surprised by the results of this experiment?" Most hands went up amidst a general murmur. One student called out, "I really thought they'd be the same!"

Kelly called from the back of the room, "Ms. Lister?"

"Yes, Kelly."

"This was really like the problem we had for homework last week. You know, the one about which pipe would carry more water—one pipe with a 3-inch diameter or three pipes with 1-inch diameters."

"Good observation, Kelly," she said. "How did that help you do this problem?"

"I figured that there would be a difference, and I was pretty sure the one with the bigger diameter would have bigger volume. Our computations showed that, too."

"Great connection!" Ms. Lister praised.

Ryan patiently waited with his hand raised. "Ryan?" Ms. Lister called.

"So does this mean that if the perimeter of one figure is the same as another, then their areas might be different, too?"

LaShauna turned to Ryan. "Yeah, remember that problem with circles and squares?"

"When we had a square with the same perimeter as a circle and we were supposed to find out which one had the greater area. It turned out that the circle did."

"Oh, yeah," said Ryan, with little enthusiasm.

"I have another question for you to think about," Ms. Lister added. "Suppose you have two plane figures. One has a really, really large perimeter—say one thousand units. The other has a much smaller perimeter, like ten units. Does the area of the figure with the larger perimeter *have to be* larger than the one with the smaller perimeter? If so, can you explain why? If not, can you come up with a counter example?"

"Is that extra credit?" Jamie interjected.

"Sure," Ms. Lister responded after a brief pause. "Up to five points on your next test."

There was a general stir in the room as students hurriedly wrote down the question.

Lucy's View

"Okay. Let's get back to our cylinders for a moment." Ms. Lister turned to Lucy. "Lucy, I thought I heard you give a good explanation of the reason for the difference in volumes to Jamie. Would you like to tell the rest of the class?"

Lucy was a diligent student who had been in Ms. Lister's class the previous 2 years. She had had some difficulty, which prompted her parents to have her tested for learning disabilities by an independent organization. The Individual Education Plan, which was developed following the evaluation, called for untimed testing and pointed out strategies to help Lucy study more efficiently. The main goal of the IEP, as her parents readily admitted, was to enable Lucy to qualify for extended testing time on the SAT. Competition for college was very stiff in this community, and even though the exam would be stamped to indicate the extra time, her parents felt that the undoubtedly higher score would be worth it.

Lucy responded, "Well, it has to do with the relationship of the radius to the height."

"Let me write on the board what you are saying," Ms. Lister said.

Lucy continued, "The volume of the tall one is *pi* times the square of the radius of the first one, times the height of the first one. The volume of the second one is *pi* times the square of the radius of the second one, times the height of the second one." Lucy paused while Ms. Lister wrote on the board:

$$\pi r_1^2 h = \pi r_2^2 h$$

"You can cancel out the *pi*'s." She paused again while Ms. Lister crossed them out. "Then you root them and you get radius times height of the first one, and radius times height of the second one, and see if they are the same."

"Wait," Ms. Lister said, sensing an unexpected problem. "What do you mean 'root' them?"

"Let me show you." Lucy got up and went to the board. "You start with this":

$$\pi r^2 h = \pi r^2 h$$

Ms. Lister interrupted. "But don't you need to make some differentiation between the radii and heights of the two cylinders? You can't use the same letters for both."

Lucy thought for a moment. "Oh, I see. I'll use small letters for the short one and large for the tall one." She then erased the righthand side and rewrote the equation as

$$\pi r^2 h = \pi R^2 H$$

"Then you can cancel the pi's, so you have":

$$r^2 h = R^2 H$$

"Then you root both sides and get":

$$rh = RH$$

Ms. Lister glanced at the clock. There was only 1 minute left in class and she saw that the other students were starting to close up their notebooks.

"Thanks, Lucy. I want to start right here tomorrow. You can go back to your seat now."

Just as the bell rang Sergio called out, "Is there a smallest and a greatest volume using the same piece of paper?"

"Great question," said Ms. Lister as students were filing out of the room. "Think about it for tomorrow!"

After the students had left, Mrs. Jenkins approached her friend, who was standing at her desk reading the papers the students had just tossed on her desk. "So, are you surprised?"

"I am amazed," Ms. Lister replied, with raised eyebrows.

QUESTIONS

1. What did Ms. Lister want her students to understand from this activity? In your view, did they? Why or why not?
2. What types of activities does Ms. Lister use in this class? Are there other approaches that might be effective that she doesn't employ? Is this inquiry teaching?
3. What would you do the next day? How would you address the issues related to Lucy; the group of Amber, Hope, and Maggie; and Sergio's question?